P-adic Measures and P-adic Spaces of Continuous Functions

A. K. Katsaras
Department of Mathematics
University of Ioannina
45110 Ioannina, Greece

Key words and phrases: Non-Archimedean fields, zero-dimensional spaces, tensor products, locally convex spaces.

1991 *Mathematics Subject Classification: 46A40*

Introduction

Let $\mathbb{K}$ be a complete non-Archimedean valued field and let $C(X, E)$ be the space of all continuous functions from a zero-dimensional Hausdorff topological space X to a non-Archimedean Hausdorff locally convex space E. We will denote by $C_b(X, E)$ (resp. by $C_{rc}(X, E)$) the space of all $f \in C(X, E)$ for which $f(X)$ is a bounded (resp. relatively compact) subset of E. The dual space of $C_{rc}(X, E)$, under the topology t_u of uniform convergence, is a space $M(X, E')$ of finitely-additive E'-valued measures on the algebra $K(X)$ of all clopen , i.e. both closed and open, subsets of X. Some subspaces of $M(X, E'$ turn out to be the duals of $C(X, E)$ or of $C_b(X, E)$ under certain locally convex topologies.
In section 2 of this paper, we give some results about the space $M(X, E')$, while in section 3 we study some of the properties of the so called Q-integrals, a concept given by the author in [14]. In section 4, we identify the dual of $C_b(X, E)$ under the strict topology β_1. The notion of a θ_0-complete topological space was given in [1]. In section 5 we study some of the properties of θ_o-complete spaces. Among other results, we prove that a Hausdorff zero-dimensional space is θ_o-complete iff it is homeomorphic to a closed subspace of a product of ultrametric spaces. In section 6, we prove that the dual space of $C(X, E)$, under the topology of uniform convergence on the bounding subsets of X, is the space of all $m \in M(X, E')$ which have a bounding support. In section 7 it is shown that the space $M_s(X)$ of all separable members of $M(X)$, under the topology of uniform convergence on the uniformly bounded equicontinuous subsets of X, is complete. The same is proved in section 8 for the space $M_{sv_o}(X)$ of those separable m for which the support of the extension m^{β_o}, to all of the Banaschewski compactification $\beta_o X$ of X, is contained in the $\mathbf{N}$-repletion $v_o X$ of X, if we equip $M_{sv_o}(X)$ with the topology

of uniform convergence on the pointwise bounded equicontinuous subsets of $C(X)$. In section 9, we give necessary and sufficient conditions for the space $C(X, E)$, equipped with the topology of uniform convergence on the compact subsets of X, to be polarly barrelled, polarly quasi-barrelled, polarly absolutely quasi-barrelled, polarly $\aleph_o$-barrelled, polarly ℓ^∞-barrelled or polarly c_o-barrelled. Finally, in section 10 , we study tensor products of spaces of continuous functions as well as tensor products of certain E'-valued measures.

1 Preliminaries

Throughout this paper, $\mathbb{K}$ will be a complete non-Archimedean valued field, whose valuation is non-trivial. By a seminorm, on a vector space over $\mathbb{K}$, we will mean a non-Archimedean seminorm. Similarly, by a locally convex space we will mean a non-Archimedean locally convex space over $\mathbb{K}$ (see [22]). Unless it is stated explicitly otherwise, X will be a Hausdorff zero-dimensional topological space , E a Hasusdorff locally convex space and $cs(E)$ the set of all continuous seminorms on E. The space of all $\mathbb{K}$-valued linear maps on E is denoted by $E^\star$, while E' denotes the topological dual of E. A seminorm p, on a vector space G over $\mathbb{K}$, is called polar if $p = \sup\{|f| : f \in G^\star, |f| \leq p\}$. A locally convex space G is called polar if its topology is generated by a family of polar seminorms. A subset A of G is called absolutely convex if $\lambda x + \mu y \in A$ whenever $x, y \in A$ and $\lambda, \mu \in \mathbb{K}$, with $|\lambda|, |\mu| \leq 1$. We will denote by $\beta_o X$ the Banaschewski compactification of X (see [5]) and by $\upsilon_o X$ the N-repletion of X, where $\mathbf{N}$ is the set of natural numbers. We will let $C(X, E)$ denote the space of all continuous E-valued functions on X and $C_b(X, E)$ (resp. $C_{rc}(X, E)$) the space of all $f \in C(X, E)$ for which $f(X)$ is a bounded (resp. relatively compact) subset of E. In case $E = \mathbb{K}$, we will simply write $C(X), C_b(X)$ and $C_{rc}(X)$ respectively. For $A \subset X$, we denote by χ_A the $\mathbb{K}$-valued characteristic function of A. Also, for $X \subset Y \subset \beta_o X$, we denote by $\bar{B}^Y$ the closure of A in Y. If $f \in E^X, p$ a seminorm on E and $A \subset X$, we define

$$\|f\|_p = \sup_{x \in X} p(f(x)), \quad \|f\|_{A,p} = \sup_{x \in A} p(f(x)).$$

The strict topology β_o on $C_b(X, E)$ (see [9]) is the locally convex topology generated by the seminorms $f \mapsto \|hf\|_p$, where $p \in cs(E)$ and h is in the space $B_o(X)$ of all bounded $\mathbb{K}$-valued functions on X which vanish at infinity, i.e. for every $\epsilon > 0$ there exists a compact subset Y of X such that $|h(x)| < \epsilon$ if $\notin Y$.
Let $\Omega = \Omega(X)$ be the family of all compact subsets of $\beta_o X \setminus X$. For $H \in \Omega$, let C_H be the space of all $h \in C_{rc}(X)$ for which the continuous extension h^{β_o} to all of $\beta_o X$ vanishes on H. For $p \in cs(E)$, let $\beta_{H,p}$ be the locally convex topology on $C_b(X, E)$ generated by the seminorms $f \mapsto \|hf\|_p, \quad h \in C_H$. For $H \in \Omega, \beta_H$ is the locally convex topology on $C_b(X, E)$ generated by the seminorms $f \mapsto \|hf\|_p, \quad h \in C_H, p \in cs(E)$. The inductive limit of the topologies $\beta_H, H \in \Omega$, is the topology β. Replacing Ω by the family Ω_1 of all $\mathbb{K}$-zero subsets of $\beta_o X$, which are disjoint from X, we get the topology β_1. Recall that a $\mathbb{K}$-zero subset of $\beta_o X$ is a set of the form $\{x \in \beta_o X : g(x) = 0\}$, for some $g \in C(\beta_o X)$. We get the topologies β_u and β'_u replacing Ω by the family Ω_u of all $Q \in \Omega$ with the following property: There

exists a clopen partition $(A_i)_{i\in I}$ of X such that Q is disjoint from each $\overline{A_i}^{\beta_o X}$. Now β_u is the inductive limit of the topologies β_Q, $Q \in \Omega_u$. The inductive limit of the topologies $\beta_{H,p}$, as H ranges over Ω_u, is denoted by $\beta_{u,p}$, while β'_u is the projective limit of the topologies $\beta_{u,p}$, $p \in cs(E)$. For the definition of the topology β_e on $C_b(X)$ we refer to [12].

Let now $K(X)$ be the algebra of all clopen subsets of X. We denote by $M(X, E')$ the space of all finitely-additive E'-additive measures m on $K(X)$ for which the set $m(K(X))$ is an equicontinuous subset of E'. For each such m, there exists a $p \in cs(E)$ such that $\|m\|_p = m_p(X) < \infty$, where, for $A \in K(X)$,

$$m_p(A) = \sup\{|m(B)s|/p(s) : p(s) \neq 0, \quad A \supset B \in K(X)\}.$$

The space of all $m \in M(X, E')$ for which $m_p(X) < \infty$ is denoted by $M_p(X, E')$. In case $E = \mathbb{K}$, we denote by $M(X)$ the space of all finitely-additive bounded $\mathbb{K}$-valued measures on $K(X)$. An element m of $M(X)$ is called τ-additive if $m(V_\delta) \to 0$ for each decreasing net (V_δ) of clopen subsets of X with $\bigcap V_\delta = \emptyset$. In this case we write $V_\delta \downarrow \emptyset$. We denote by $M_\tau(X)$ the space of all τ-additive members of $M(X)$. Analogously, we denote by $M_\sigma(X)$ the space of all σ-additive m, i.e. those m with $m(V_n) \to 0$ when $V_n \downarrow \emptyset$. For an $m \in M(X, E')$ and $s \in E$, we denote by ms the element of $M(X)$ defined by $(ms)(V) = m(V)s$.

Next we recall the definition of the integral of an $f \in E^X$ with respect to an $m \in M(X, E')$. For a non-empty clopen subset A of X, let $\mathcal{D}_A$ be the family of all $\alpha = \{A_1, A_2, \ldots, A_n; x_1, x_2, \ldots, x_n\}$, where $\{A_1, \ldots, A_n\}$ is a clopen partition of A and $x_k \in A_k$. We make $\mathcal{D}_A$ into a directed set by defining $\alpha_1 \geq \alpha_2$ iff the partition of A in α_1 is a refinement of the one in α_2. For an $\alpha = \{A_1, A_2, \ldots, A_n; x_1, x_2, \ldots, x_n\} \in \mathcal{D}_A$ and $m \in M(X, E')$, we define

$$\omega_\alpha(f, m) = \sum_{k=1}^{n} m(A_k) f(x_k).$$

If the limit $\lim \omega_\alpha(f, m)$ exists in $\mathbb{K}$, we will say that f is m-integrable over A and denote this limit by $\int_A f\, dm$. We define the integral over the empty set to be 0. For $A = X$, we write simply $\int f\, dm$. It is easy to see that if f is m-integrable over X, then it is integrable over every clopen subset A of X and $\int_A f\, dm = \int \chi_A f\, dm$. If τ_u is the topology of uniform convergence, then every $m \in M(X, E')$ defines a τ_u-continuous linear functional ϕ_m on $C_{rc}(X, E)$, $\phi_m(f) = \int f\, dm$. Also every $\phi \in (C_{rc}(X, E), \tau_u)'$ is given in this way by some $m \in M(X, E')$.

For $p \in cs(E)$, we denote by $M_{t,p}(X, E')$ the space of all $m \in M_p(X, E')$ for which m_p is tight, i.e. for each $\epsilon > 0$, there exists a compact subset Y of X such that $m_p(A) < \epsilon$ if the clopen set A is disjoint from Y. Let

$$M_t(X, E') = \bigcup_{p \in cs(E} M_{t,p}(X, E').$$

Every $m \in M_{t,p}(X, E')$ defines a β_0-continuous linear functional u_m on $C_b(X, E)$, $u_m(f) = \int f\, dm$. The map $m \mapsto u_m$, from $M_t(X, E')$ to $(C_b(X, E), \beta_o)'$, is an algebraic isomorphism. For $m \in M_\tau(X)$ and $f \in \mathbb{K}^X$, we will denote by $(VR) \int f\, dm$

the integral of f , with respect to m, as it is defined in [22]. We will call $(VR)\int f\,dm$ the (VR)-integral of f.

For all unexplained terms on locally convex spaces, we refer to [21] and [22].

2 Some results on $M(X, E')$

Theorem 2.1 *Let $m \in M(X, E')$ be such that $ms \in M_\tau(X)$, for all $s \in E$, and let $p \in cs(E$ with $\|m\|_p < \infty$. Then :*

1. *$m_p(V) = \sup_{x\in V} N_{m,p}(x)$ for every $V \in K(X)$.*
2. *The set*
$$supp(m) = \bigcap\{V \in K(X) : m_p(V^c) = 0\}$$
is the smallest of all closed support sets for m.
3. *$supp(m) = \overline{\{x : N_{m,p}(x) \neq 0\}}$.*
4. *If V is a clopen set contained in the union of a family $(V_i)_{i\in I}$ of clopen sets, then*
$$m_p(V) \leq \sup\{m_p V_i) : i \in I\}.$$

Proof: (1). If $x \in V$, then $N_{m,p}(x) \leq m_p(V)$ and so

$$m_p(V) \geq \alpha = \sup_{x\in V} N_{m,p}(x).$$

On the other hand, let $m_p(V) > d$. There exists a clopen set W, contained in V, and $s \in E$ with $|m(W)s|/p(s) > d$. Let $\mu = ms \in M_\tau(X)$. Then

$$|\mu|(W) = \sup_{x\in W} N_\mu(x).$$

Let $x \in W$ be such that $N_\mu(x) > d \cdot p(s)$. Now $N_{m,p}(x) \geq d$. In fact, assume the contrary and let Z be a clopen neighborhood of x contained in W and such that $m_p(Z) < d$. Now

$$N_\mu(x) \leq |\mu|(Z) = \sup\{|m(Y)s| : Z \supset Y \in K(X)\} \leq p(s) \cdot m_p(Z) \leq d \cdot p(s).$$

This contradiction proves (1).

(2).
$$X \setminus supp(m) = \bigcup\{W \in K(X) \colon m_p(W) = 0\}.$$

Let $V \in K(X)$ be disjoint from $supp(m)$. For each $x \in V$, there exists $W \in K(X)$, with $x \in W$ and $m_p(W) = 0$ and so $N_{m,p}(x) = 0$. It follows that

$$m_p(V) = \sup_{x\in V} N_{m,p}(x) = 0,$$

which proves that $supp(m)$ is a support set for m. On the other hand, let Y be a closed support set for m. There exists a decreasing net (V_δ) of clopen sets with $Y = \bigcap V_\delta$. Let $W \in K(X)$ be disjoint from Y. For each clopen set V contained in W and each $s \in E$, we have $V \cap V_\delta \downarrow \emptyset$ and so $\lim_\delta (ms)(V \cap V_\delta) = 0$. Since V_δ^c is disjoint from Y, we have $m(V_\delta^c) = 0$ and so $m(V) = m(V_\delta \cap V)$, which implies that $m(V)s = 0$, for all $s \in E$, i.e. $m(V) = 0$, and hence $m_p(W) = 0$. Therefore $supp(m) \subset W^c$. Taking V_δ^c in place of W, we get that $supp(m) \subset \bigcap V_\delta = Y$, which proves (2).

(3) Let $G = \overline{x : N_{m,p}(x) \neq 0}\}$. If $V \in K(X)$ is disjoint from G, then

$$m_p(V) = \sup_{x \in V} N_{m,p}(x) = 0,$$

and so $supp(m) \subset V^c$, which implies that $supp(m) \subset G$. On the other hand, let $x \notin supp(m)$. There exists a clopen neighborhood W of x disjoint from $supp(m)$. Since $supp(m)$ is a support set for m, we have that $m_p(W) = 0$ and thus $N_{m,p} = 0$ on W, which proves that $x \notin G$. Thus G is contained in $supp(m)$ and (3) follows.

(4). Let $m_p(V) > \alpha > 0$. There exists a clopen set A contained in V and $s \in E$ such that $|m(A)s|/p(s) > \alpha$. If $\mu = ms \in M_\tau(X)$, then $|\mu|(V) \geq |m(A)s| > \alpha \cdot p(s)$. In view of [22], p. 250 , there exists an i such that $m_p(V_i) \geq |\mu|(V_i)/p(s) > \alpha$, which clearly completes the proof.

Theorem 2.2 *Let $m \in M(X, E')$ be such that $ms \in M_\sigma(X)$ for alll $s \in E$ (this in particular holds if $m \in M_\sigma(X, E')$). Let $p \in cs(E)$ be such that $m_p(X) < \infty$. If a clopen set V is contained in the union of a sequence (V_n) of clopen sets, then $m_p(V) \leq \sup_n m_p(V_n)$.*

Proof : We show first that, for $\mu \in M_\sigma(X)$, then there exists an n with $|\mu|(V) \leq |\mu|(V_n)$. In fact, this is clearly true if $|\mu|(V) = 0$. Assume that $|\mu|(V) > 0$ and let $W_n = \bigcup_1^N V_k$. Since $W_n^c \cap V \downarrow \emptyset$, there exists n such that $|\mu|(V \cap W_n^c) < |\mu|(V)$. Since $V \subset (V \bigcap W_n^c) \bigcup W_n$, it follows that

$$|\mu|(V) \leq |\mu|(W_n) = \max_{1 \leq k \leq n} |\mu|(V_k),$$

and the claim follows for μ. Suppose now that $m_p(V) > r > 0$. There exists a clopen subset W of V and $s \in E$ such that $|m(W)s| > r \cdot p(s)$. Let $\mu = ms$. Then $\mu \in M_\sigma(X)$ and $|\mu|(V) \geq |m(W)s| > r \cdot p(s)$. By the first part of the proof, there exists an n such that $|\mu|(V_n) > r \cdot p(s)$. Hence, there exists a clopen subset D of V_n such that $|\mu(D)| > r \cdot p(s)$. Now $|m|_p(V_n) \geq |m(D)s|/p(s) > r$, which completes the proof.

For $X \subset Y \subset \beta_o X$, and $m \in M(X)$, we denote by m^Y the element of $M(Y)$ defined by $m^Y(V) = m(V \cap X)$. We denote by m^{v_o} and m^{β_o} the m^Y for $Y = v_o X$ and $Y = \beta_o X$, respectively.

We have the following easily established

Theorem 2.3 *Let $m \in M(X, E')$ be such that $ms \in M_\tau(X)$ for all $s \in E$. Then :*

1. $supp(m^{\beta_o}) = \overline{supp(m)}^{\beta_o X}$.

2. $supp(m) = supp(m^{\beta_o}) \cap X$.

3. *If m has compact support, then* $supp(m) = supp(m^{\beta_o})$.

Theorem 2.4 *For an $m \in M(X)$, the following are equivalent:*

1. *$supp(m^{\beta_o}) \subset \upsilon_o X$.*

2. *If (V_n) is a sequence of clopen subsets of X which decreases to the empty set, then there exists n with $|m|(V_n) = 0$.*

3. *If $V_n \downarrow \emptyset$, then there exists an n_o such that $m(V_n) = 0$ for all $n \geq n_o$.*

4. *If $Z \in \Omega_1$, then there exists a clopen subset A of $\beta_o X$, containing Z, with $|m^{\beta_o}|(A) = 0$.*

Proof : (1) $\Rightarrow$ (3). If $V_n \downarrow \emptyset$, then $\bigcap \overline{V_n}^{\beta_o X}$ is disjoint from $\upsilon_o X$ and thus $supp(m^{\beta_o}) \subset \bigcup_n \overline{V_n^c}^{\beta_o X}$. In view of the compactness of $supp(m^{\beta_o}$, there exists an n_o such that $supp(m^{\beta_o}) \subset \overline{V_{n_o}^c}^{\beta_o X}$. Thus, for $n \geq n_o$, we have $m(V_n) = m^{\beta_o}(\overline{V_n}^{\beta_o X}) = 0$.
(3) $\Rightarrow$ (2). Let $V_n \downarrow \emptyset$ and suppose that $|m|(V_n) > 0$ for all n.
Claim: For each n, there exists $k > n$ and a clopen set B with $V_k \subset B \subset V_n$, with $m(B) \neq 0$. Indeed, there exists a clopen subset A of V_n with $m(A) \neq 0$. For each k, let $B_k = V_k \cap A, D_k = V_k \setminus B_k$. Then $D_k \downarrow \emptyset$. By our hypothesis, there exists $k > n$ such that $m(D_k) = 0$. Let $B = A \cup D_k$. The sets A and D_k are disjoint and so $m(B) = m(A) \neq 0$. Moreover $V_k \subset B \subset V_n$, which proves our claim.
We choose now inductively $n_1 = 1 < n_2 < \ldots$ and clopen sets $B_1, B_2, \ldots$, with $V_{n_{k+1}} \subset B_k \subset V_{n_k}$, $m(B_k) \neq 0$. This is a contradiction since $B_k \downarrow \emptyset$.
(2) $\Rightarrow$ (4). Let $Z \in \Omega_1$. There exists a sequence (V_n) of clopen subsets of X such that $V_n \downarrow \emptyset$ and $\bigcap \overline{V_n}^{\beta_o X} = Z$. By our hypothesis, there exists an n such that $|m|(V_n) = 0$. Now it suffices to take $A = \overline{V_n}^{\beta_o X}$.
(4) $\Rightarrow$ (2). Let $V_n \downarrow \emptyset$ and take $W_n = \overline{V_n}^{\beta_o X}, Z = \cap W_n$. By our hypothesis, there exists a clopen subset A of $\beta_o X$ containing Z with $|m^{\beta_o}|(A) = 0$. Now $A^c \subset W_n^c$ for some n. Thus $|m|(V_n) = |m^{\beta_o}|(W_n) = 0$.
(2) $\Rightarrow$ (1). Suppose that there exists $z \in supp(m_o^{\beta})$ which is not in $\upsilon_o X$. Then there exists a sequence (V_n) of clopen subsets of X with $V_n \downarrow \emptyset$ and $z \in \overline{V_n}^{\beta_o X}$ for all n. By our hypothesis, there exists an n with $|m^{\beta_o}|(\overline{V_n}^{\beta_o X}) = |m|(V_n) = 0$, which is a contradiction since $z \in supp(m^{\beta_o})$. Hence the result follows.

Theorem 2.5 *For an $m \in M(X)$, the following are equivalent :*

1. *m has compact support, i.e. $m \in M_c(X)$.*

2. *$supp(m^{\beta_o}) \subset X$.*

3. *If $V_\delta \downarrow \emptyset$, then there exists a δ_o such that $m(V_\delta) = 0$ for all $\delta \geq \delta_o$.,*

4. *If $V_\delta \downarrow \emptyset$, then $|m|(V_\delta) = 0$ for some δ.*

5. *If $H \in \Omega$, then there exists a clopen subset A of $\beta_o X$, containing H, with $|m^{\beta_o}|(A) = 0$.*

Proof : In view of Theorem 2.3, (1) implies (2).
(2) $\Rightarrow$ (3). Let $V_\delta \downarrow \emptyset$. By the compactness of $supp(m_o^\beta)$, there exists δ_o such that $supp(m^{\beta_o}) \subset V_{\delta_o}^c$ and so $m(V_\delta) = 0$ for $\delta \geq \delta_o$.
(3) $\Rightarrow$ (4). Let $V_\delta \downarrow \emptyset$ and suppose that $|m|(V_\delta) > 0$ for al δ.
Claim. For each δ there exist $\gamma \geq \delta$ and a clopen set A such that $V_\gamma \subset A \subset V_\delta$ and $m(A) \neq 0$. In fact, there exists a clopen subset G of V_δ with $m(G) \neq 0$. For each γ, let $Z_\gamma = V_\gamma \cap G, W_\gamma = V_\gamma \setminus Z_\gamma$. Then $W_\gamma \downarrow \emptyset$. By our hypothesis, there exists $\gamma \geq \delta$ with $m(V_\gamma) = 0$. Let $A = G \cup W_\gamma$. Since the sets G and W_γ are disjoint, we have that $m(A) = m(G) \neq 0$. Since $V_\gamma \subset A \subset V_\delta$, the claim follows.
Let now $\mathcal{F}$ be the family of all clopen subsets A of X with the following property: There are γ, δ, with $\gamma \geq \delta$, $V_\gamma \subset A \subset V_\delta$ and $m(A) \neq 0$. Since $\mathcal{F} \downarrow \emptyset$, we got a contradiction.
(4) $\Rightarrow$ (5). If $H \in \Omega$, then there exists a decreasing net (V_δ) of clopen subsets of X with $\bigcap \overline{V_\delta}^{\beta_o X} = H$. Since $V_\delta \downarrow \emptyset$, there exists δ such that $|m|(V_\delta) = 0$. Now it suffices to take $A = \overline{V_\delta}^{\beta_o X}$.
(5) $\Rightarrow$ (2). Suppose that there exists a $z \in supp(m^{\beta_o})$ which is not in X. Then there exists a decreasing net (V_δ) of clopen subsets of X with $\bigcap \overline{V_\delta}^{\beta_o X} = \{z\}$. Using (5), we get that there exists a δ such that $|m^{\beta_o}|(\overline{V_\delta}^{\beta_o X}) = 0$, which is a contradiction since $z \in supp(m^{\beta_o})$.
(2) $\Rightarrow$ (1). It is trivial since $supp(m^{\beta_o})$ is a support set for m.

Theorem 2.6 *For an $m \in M_\sigma(X)$, the following are equivalent :*

1. *$m \in M_s(X)$.*
2. *For each continuous ultrapseudometric d on X, there exists a d-closed, d-separable subset G of X such that $m(V) = 0$ for each d-clopen set V disjoint from G.*

Proof: (1) $\Rightarrow$ (2). Let d be a continuous ultrapseudometric on X and let $\mu = T_d^\star m \in M_\tau(X_d)$. By [12], Theorem 6.2, there exists a closed separable subset Z of X_d such that $|\mu|^*(X_d \setminus Z) = 0$. If $z \in X_d \setminus Z$, then $N_\mu(z) = 0$. In fact, given $\epsilon >$, there is a sequence (A_n) of clopen subsets of X_d covering $X_d \setminus Z$ and $\sup_n |\mu|(A_n) < \epsilon$ and so $N_\mu(z) < \epsilon$. If now B is a clopen subset of X_d disjoint from Z, then $|\mu|(B) = \sup_{z \in B} N_\mu(z) = 0$. If $G = \pi_d^{-1}(Z)$, then G is d-closed, d-separable and $m(V) = 0$ for each d-clopen set V disjoint from G.
(2) $\Rightarrow$ (1). Let $(V_i)_{i \in I}$ be a clopen partition of X and let $f_i = \chi_{V_i}$. Define

$$d(x, y) = \sup_i |f_i(x) - f_i(y)|.$$

Then, d is a continuous ultrapseudometric on X. Each V_i is d-clopen and hence $\bigcup_{i \in J} V_i$ is d-clopen for each subset J of I. Since G is d-separable (and hence d-Lindelöf), there exists a countable subset $J = \{i_1, i_2, \ldots\}$ such that $G \subset \bigcup_k V_{i_k}$. Let $J_1 = I \setminus J$. The set $V = \bigcup_{i \in J_1} V_i$ is d-clopen and $m(V) = 0$. Also, $m(V_i) = 0$ for $i \in J_1$. Since m is σ-additive, we have that

$$m(X) = m(V) + \sum_{k=1}^{\infty} m(V_{i_k}) = \sum_{k=1}^{\infty} m(V_{i_k}) = \sum_{i \in I} m(V_i).$$

This (In view of [12], Theorem 6.9) proves that $m \in M_s(X)$ and the result follows.

3 Q-Integrals

We will recall next the definition of the Q-integral which was given in [14]. Let $m \in M(X, E')$ be such that $ms \in M_\tau(X)$ for all $s \in E$. This in particular happens if $m \in M_\tau(X, E')$. For $f \in E^X$ and $x \in X$, we define

$$Q_{m,f}(x) = \inf_{x \in V \in K(X)} \sup\{|m(B)f(x)| : V \supset B \in K(X)\}, \quad \|f\|_{Q_m} = \sup_{x \in X} Q_{m,f}(x).$$

Let $S(X, E)$ be the linear subspace of E^X spanned by the functions $\chi_A s,\ s \in E,\ A \in K(X)$, where χ_A is the $\mathbb{K}$-characteristic function of A. We will write simply $S(X)$ if $E = \mathbb{K}$.

Lemma 3.1 *If $g \in S(X, E)$, then*

$$\|g\|_{Q_m} = \sup_{x \in X} Q_{m,g}(x) < \infty.$$

Proof: The proof was given in [14], Lemma 7.2. Note that, if $\|m\|_p < \infty$ and $d \geq \|g\|_p$, then $Q_{m,g}(x) \leq d \cdot m_p(X)$.

Lemma 3.2 *For $g \in S(X, E)$, we have*

$$\left| \int g \, dm \right| \leq \|g\|_{Q_m}.$$

Proof: Assume first that $g = \chi_A s,\ A \in K(X)$. For $x \in A$, we have

$$|m(A)s| \leq |ms|(A) = \sup_{y \in A} N_{ms}(y).$$

But, for $y \in A$, we have

$$N_{ms}(y) = \inf_{y \in V \in K(X)} \sup_{V \supset B \in K(X)} |m(B)s| = \inf_{y \in V \in K(X)} \sup_{V \supset B \in K(X)} |m(B)g(y)| = Q_{m,g}(y).$$

Thus $|m(A)s| \leq \sup_{y \in A} Q_{m,g}(y)$. In the general case, there are pairwise disjoint clopen sets $A_1, \ldots, A_n$ covering X and $s_k \in E$ with $g = \sum_{k=1}^n \chi_{A_k} s_k$. Thus,

$$\left| \int g \, dm \right| = \left| \sum_{k=1}^n m(A_k) s_k \right| \leq \max_{1 \leq k \leq n} |m(A_k) s_k| \leq \sup_{x \in X} Q_{m,g}(x) = \|g\|_{Q_m}.$$

Definition 3.3 *Let $m \in M(X, E')$ be such that $ms \in M_\tau(X)$ for all $s \in E$. A function $f \in E^X$ is said to be Q-integrable with respect to m if there exists a sequence (g_n) in $S(X, E)$ such that $\|f - g_n\|_{Q_m} \to 0$. In this case, the Q-integral of f is defined by*

$$(Q) \int f \, dm = \lim_{n \to \infty} \int g_n \, dm.$$

If f is Q-integrable with respect to m, then for $A \in K(X)$ the function $\chi_A f$ is also Q-integrable. We define

$$(Q)\int_A f\,dm = (Q)\int \chi_A f\,dm.$$

As it is proved in [14], the Q-integral is well defined. If $\mu \in M_\tau(X)$ and $g \in \mathbb{K}^X$, then $Q_{\mu,g}(x) = |g(x)|N_\mu(x)$. Thus the Q-integral with respect to μ coincides with the integral as it is defined in [22], which we will call (VR)-integral. Hence

$$(VR)\int g\,d\mu = (Q)\int g\,d\mu.$$

Lemma 3.4 *If $f \in E^X$ is Q-integrable with respect to an $m \in M(X, E')$ and if (g_n) is a sequence in $S(X, E)$, with $\|f - g_n\|_{Q_m} \to 0$, then*

$$\|f\|_{Q_m} = \lim_{n\to\infty} \|g_n\|_{Q_m} < \infty, \quad \textit{and} \quad \left|(Q)\int f\,dm\right| \le \|f\|_{Q_m}.$$

Proof: Since

$$Q_{m,h+g}(x) \le \max\{Q_{m,g}(x), Q_{m,h}(x)\},$$

it follows that

$$\|h + g\|_{Q_m} \le \max\{\|h\|_{Q_m}, \|g\|_{Q_m}\}.$$

Thus

$$\|f\|_{Q_m} \le \max\{\|f - g_n\|_{Q_m}, \|g_n\|_{Q_m}\} \le \|f - g_n\|_{Q_m} + \|g_n\|_{Q_m} < \infty.$$

It follows that

$$|\|f\|_{Q_m} - \|g_n\|_{Q_m}| \le \|f - g_n\|_{Q_m} \to 0.$$

Moreover,

$$\left|(Q)\int f\,dm\right| = \lim_{n\to\infty}\left|\int g_n\,dm\right| \le \lim_{n\to\infty}\|g_n\|_{Q_m} = \|f\|_{Q_m}.$$

Hence the result follows.

Theorem 3.5 *Let $m \in M(X, E')$ be such that $ms \in M_\tau(X)$ for all $s \in E$, and let $f \in E^X$ be Q-integrable. Define*

$$m_f : K(X) \to \mathbb{K}, \quad m_f(A) = (Q)\int_A f\,dm.$$

Then $m_f \in M_\tau(X)$.

Proof: Since $|m_f(A)| \le \|f\|_{Q_m}$, it is easy to see that $m_f \in M(X)$. Let now $V_\delta \downarrow \emptyset$ and $\epsilon > 0$. Choose a $g = \sum_{k=1}^n \chi_{A_k} s_k \in S(X, E)$ such that $\|f - g\|_{Q_m} < \epsilon$. Then

$$\int_{V_\delta} g\,dm = \sum_{k=1}^n (ms_k)(V_\delta \cap A_k) \to 0.$$

Let δ_o be such that $\left|\int_{V_\delta} g\, dm\right| < \epsilon$ if $\delta \geq \delta_o$. Now, for $\delta \geq \delta_o$, we have

$$\begin{aligned}\left|(Q)\int_{V_\delta} f\, dm\right| &\leq \max\{\left|(Q)\int_{V_\delta}(f-g)\, dm\right|, \left|\int_{V_\delta} g\, dm\right|\} \\ &\leq \max\{\|f-g\|_{Q_m}, \left|\int_{V_\delta} g\, dm\right|\} < \epsilon.\end{aligned}$$

Thus $m_f(V_\delta) \to 0$.

Lemma 3.6 *If $f \in E^X$ is Q-integrable with respect to an $m \in M(X, E')$, then the map $x \to Q_{m,f}(x)$ is upper semicontinuous.*

Proof: We need to show that, for each $\alpha > 0$, the set

$$V = \{x : Q_{m,f}(x) < \alpha\}$$

is open. So let $x \in V$ and choose $\epsilon > 0$ such that $Q_{m,f}(x) < \alpha - 2\epsilon$. Let $g \in S(X, E)$ be such that $\|f-g\|_{Q_m} < \epsilon$. Let $A_1, \ldots, A_n$ be a clopen partition of X and $s_k \in E$ such that $g = \sum_{k=1}^n \chi_{A_k} s_k$. Let k be such that $x \in A_k$. There exists a clopen set B, containing x and contained in A_k, such that $|m(D)g(x)| < Q_{m,g}(x) + \epsilon$ for every clopen set D contained in B. If $y \in B$, then for $B \supset D \in K(X)$ we have

$$\begin{aligned}|m(D)g(y)| &= |m(D)g(x)| < Q_{m,g}(X) + \epsilon \\ &\leq \max\{Q_{m,g-f}(x), \quad Q_{m,f}(x)\} + \epsilon \\ &\leq Q_{m,f}(x) + 2\epsilon.\end{aligned}$$

Thus $Q_{m,g}(y) \leq Q_{m,f}(x) + 2\epsilon < \alpha$. Hence $x \in B \subset V$ and the result follows.

Lemma 3.7 *If $f \in E^X$ is Q-integrable with respect to an $m \in M(X, E')$, then $N_{m_f} \leq Q_{m,f}$.*

Proof: Let $x \in X$ and $\epsilon > 0$. In view of the preceding Lemma, there exists a clopen neighborhood V of X such that $Q_{m,f}(y) \leq Q_{m,f}(x) + \epsilon$ for all $y \in V$. If $V \supset B \in K(X)$, then

$$|m_f(B)| \leq \sup_{y \in B} Q_{m,f}(y) \leq Q_{m,f}(x) + \epsilon$$

and so

$$N_{m_f}(x) \leq |m_f|(V) \leq Q_{m,f}(x) + \epsilon.$$

Hence the result follows.

Lemma 3.8 *Let $m \in M(X, E')$ be such that $ms \in M_\tau(X)$ for all $s \in E$. If $g \in S(X, E)$, then $Q_{m,g} = N_{m_g}$.*

Proof: Let $\{A_1, \ldots, A_n\}$ be a clopen partition of X and $s_k \in E$ such that $g = \sum_{k=1}^n \chi_{A_k} s_k$. Suppose that $N_{m_g}(x) < \alpha$. Then, there exists a clopen neighborhood

V of x such that $|m_g|(V) < \alpha$. Let $x \in A_k$. If B is a clopen set contained in $A_k \cap V$, then

$$m_g(B) = (Q)\int_B g\,dm = \int_B g\,dm = m(B)g(x)$$

since $g = g(x)$ on B. Thus

$$Q_{m,g}(x) \leq \sup_{B \subset A_k \cap V} |m(B)g(x)| \leq |m_g|(V) < \alpha.$$

This proves that $Q_{m,g} \leq N_{m_g}$ and the result follows.

Theorem 3.9 *If $f \in E^X$ is Q-integrable with respect to an $m \in M(X, E')$, then $Q_{m,f} = N_{m_f}$.*

Proof: Assume that $N_{m_f}(x) < \alpha$ and let $0 < \epsilon < \alpha$. There exists a clopen neighborhood V of x such that $|m_f|(V) < \alpha$. Let $g \in S(X, E)$ be such that $\|f - g\|_{Q_m} < \epsilon$. For A clopen contained in V, we have

$$|m_f(A) - m_g(A)| = \left|(Q)\int (f - g)\,dm\right| \leq \|f - g\|_{Q_m} < \epsilon$$

and so

$$|m_g(A)| \leq \max\{\epsilon, |m_f(A)|\} < \alpha.$$

Thus

$$Q_{m,g}(x) = N_{m_g}(x) \leq |m_g|(V) \leq \alpha.$$

Now

$$Q_{m,f}(x) \leq \max\{Q_{m,f-g}(x),\ \ Q_{m,g}(x)\} \leq \alpha,$$

which proves that $Q_{m,f} \leq N_{m_f}$ and the result follows by Lemma 3.7.

Theorem 3.10 *Let $m \in M(X, E')$ be such that $ms \in M_\tau(X)$, for all $s \in E$, and let $f \in E^X$ be Q-integrable with respect to m. If $g \in \mathbb{K}^X$ is Q-integrable with respect to m_f, then gf is Q-integrable with respect to m and*

$$(Q)\int gf\,dm = (Q)\int g\,dm_f.$$

Proof: If $h \in \mathbb{K}^X$, then

$$Q_{m,hf}(x) = |h(x)|Q_{m,f}(x) = |h(x)|N_{m_f}(x) = Q_{m_f,h}(x).$$

Let (g_n) be a sequence in $S(X)$ such that $\|g - g_n\|_{Q_{m_f}} \to 0$. We have

$$\begin{aligned} \|g - g_n\|_{Q_{m_f}} &= \sup_{x \in X} |g(x) - g_n(x)| \cdot N_{m_f}(x) \\ &= \sup_{x \in X} Q_{m,(g-g_n)f}(x) = \|gf - g_nf\|_{Q_m}. \end{aligned}$$

If $A \in K(X)$, then $\chi_A f$ is Q-integrable with respect to m and

$$(Q)\int \chi_A f\,dm = (Q)\int_A f\,dm = m_f(A) = \int \chi_A\,dm_f.$$

It follows that, for all n, $g_n f$ is Q-integrable with respect to m and

$$(Q)\int g_n f\,dm = \int g_n\,dm_f \to (Q)\int g\,dm_f.$$

Since $g_n f$ is Q-integrable with respect to m and $\|gf - g_n f\|_{Q_m} \to 0$, it follows that gf is Q-integrable and

$$(Q)\int gf\,dm = \lim_{n\to\infty}(Q)\int g_n f\,dm = \lim_{n\to\infty}\int g_n\,dm_f = (Q)\int g\,dm_f,$$

which completes the proof.

Theorem 3.11 *Let $m \in M(X, E')$ be such that $ms \in M_\tau(X)$, for all $s \in E$, and let $p \in cs(E)$ with $\|m\|_p < \infty$. If $f \in E^X$ is Q-integrable with respect to m, then, given $\epsilon > 0$, there exists $\alpha > 0$ such that $|(Q)\int_A f\,dm| < \epsilon$ if $m_p(A) < \alpha$..*

Proof: Let $g \in S(X, E)$ with $\|f - g\|_{Q_m} < \epsilon$. For a clopen set A, we have $|\int_A g\,dm| \leq \|g\|_p \cdot m_p(A)$. Let $\alpha > 0$ be such that $\alpha \cdot \|g\|_p < \epsilon$. If $m_p(A) < \alpha$, then

$$\begin{aligned}\left|(Q)\int_A f\,dm\right| &\leq \max\{\left|(Q)\int_A (f-g)\,dm\right|, \ \left|\int_A g\,dm\right|\} \\ &\leq \max\{\|f-g\|_{Q_m}, \ \|g\|_p \cdot m_p(A)\} < \epsilon.\end{aligned}$$

Lemma 3.12 *Let $m \in M_\tau(X)$ and let $g \in \mathbb{K}^X$ be (VR)-integrable. Then, given $\epsilon > 0$, there exists $\delta > 0$ such that $\|g\|_{A,N_m} \leq \epsilon$ if $|m|(A) < \delta$.*

Proof: There exists $h \in S(X)$ such that $\|g - h\|_{N_m} \leq \epsilon$. It suffices to choose $\delta > 0$ such that $\delta \cdot \|h\| < \epsilon$.

Let $m \in M(X)$. For $A \subset X$, we define

$$|m|^\wedge(A) = \inf\{|m|(V) : V \in K(X), \quad A \subset V\}.$$

Recall that a sequence (g_n) in $\mathbb{K}^X$ converges in measure to an $f \in \mathbb{K}^X$, with respect to m (see [14], Definition 2.12) if, for each $\alpha > 0$, we have

$$\lim_{n\to\infty} |m|^\wedge\{x : |g_n(x) - g(x)| \geq \alpha\} = 0.$$

Theorem 3.13 *Let $m \in M_\tau(X)$ and let (f_n) be a sequence of (VR)-integrable, with respect to m, functions, which converges in measure to some $f \in \mathbb{K}^X$. If there exists a (VR)-integrable function $g \in \mathbb{K}^X$ such that $|f_n| \leq |g|$ for all n, then f is (VR)-integrable and*

$$(VR)\int f\,dm = \lim_{n\to\infty}(VR)\int f_n\,dm.$$

Proof: Let $\epsilon > 0$ and choose inductively $n_1 < n_2 < \ldots$ such that $|m|^\wedge(A_k) < 1/k$, where

$$A_k = \{x : |f_{n_k}(x) - f(x) \geq 1/k\}.$$

Let $V = \bigcap_{N=1}^{\infty} \bigcup_{k \geq N} V_k$. If $x \in V$, then $N_m(x) = 0$. Indeed, for every N, there exists $k \geq N$ with $x \in V_k$ and so $N_m(x) \leq |m|(V_k) < 1/k \leq 1/N$, which proves that $N_m(x) = 0$. Also, for $x \in X \setminus V$, we have $f(x) = \lim_{k\to\infty} f_{n_k}(x)$. In fact, there eists N such that $x \notin V_k$ for $k \geq N$ and so $|f_{n_k}(x) - f(x)| < 1/k \to 0$. It follows that $|f(x)| \leq |g(x)|$ when $x \notin V$. Since g is (VR)-integrable, there exists (by the preceding Lemma) $\delta > 0$ such that $\|g\|_{A.N_m} < \epsilon$ if $|m|(A) < \delta$. Let now $\alpha > 0$ be such that $\alpha \cdot \|m\| < \epsilon$. For each n, let

$$G_n = \{x : |f_n(x) - f(x)| \geq \alpha\}$$

and choose a clopen set W_n containing G_n with $|m|(W_n) < 1/n + |m|^\wedge(G_n)$. Since $|m|^\wedge(G_n) \to 0$, there exists n_o such that $|m|(W_n) < \delta$ if $n \geq n_o$. Let now $n \geq n_o$ and $x \in X$. If $x \in V$, then $N_m(x) = 0$. Suppose that $x \notin V$. Then $|f(x)| \leq |g(x)|$ and so

$$|f(x) - f_n(x)|N_m(x) \leq |g(x)|N_m(x).$$

If $x \in W_n$, then $|g(x)|N_m(x) \leq \epsilon$, since $|m|(W_n) < \delta$, while for $x \notin W_n$ we have

$$|f(x) - f_n(x)|N_m(x) \leq \alpha \cdot \|m\| < \epsilon.$$

Thus, for $n \geq n_o$, we have $\|f - f_n\|_{N_m} \leq \epsilon$. Since f_n is (VR)-integrable, it follows that f is (VR)-integrable and

$$(VR)\int f\,dm = \lim_{n\to\infty} (VR)\int f_n\,dm$$

since

$$\left|(VR)\int (f - f_n)\,dm\right| \leq \|f - f_n\|_{N_m} \to 0.$$

This completes the proof.

Let now τ be the topology of X and let $K_c(X)$ be the collection of all subsets A of X such that $A \cap Y$ is clopen in Y for each compact subset Y of X. It is easy to see that if A, A_1, A_2 are in $K_c(X)$, then each of the sets A^c, $A_1 \cap A_2$ and $A_1 \cup A_2$ is also in $K_c(X)$. Now $K_c(X)$ is a base for a zero-dimensional topology τ^k on X finer than τ. We will denote by $X^{(k)}$ the set X equipped with the topology τ^k. We have the following easily established

Theorem 3.14 *1. τ and τ^k have the same compact sets.*

2. τ and τ^k induce the same topology on each τ-compact subset of X.

3. A subset B of X is τ^k-clopen iff $B \in K_c(X)$.

4. If Y is a zero-dimensional topological space and $f : X \to Y$, then f is τ^k-continuous iff f the restriction of f to every compact subset of X is τ-continuous.

Let now $m \in M(X, E')$ be such that $ms \in M_\tau(X)$ for each $s \in E$.

Lemma 3.15 *If $B \in K_c(X)$, $s \in E$ and $h = \chi_B s$, then h is Q-integrable with respect to m.*

Proof: Let $\epsilon > 0$. Since $ms \in M_\tau(X)$, there exists a compact subset Y of X such that $|ms|(V) < \epsilon$ for each clopen subset V of X disjoint from Y. Let $A \in K(X)$, with $B \cap Y = A \cap Y$, and let $g = \chi_A s$, $f = h - g$. If $x \in A \Delta B$, then x is not in Y and so there exists $V \in K(X)$ such that $x \in V \subset Y^c$. If $W \in K(X)$ is contained in V, then $|m(W)f(x)| = |m(W)s| \leq |ms|(V) < \epsilon$ and so $Q_{m,f}(x) \leq \epsilon$. Thus $\|h - g\|_{Q_m} \leq \epsilon$. Hence the Lemma follows.

Now for $B \in K_c(X)$, we define

$$m^{(k)}(B) : E \to \mathbb{K}, \quad m^{(k)}(B)s = (Q)\int \chi_B s \, dm.$$

Clearly $m^{(k)}$ is linear. Let $p \in cs(E)$ be such that $m_p(X) < \infty$.

Theorem 3.16 *Let $A \in K_c(X)$, and let $V \in K(X)$ with $A \subset V$. Then :*

1. *$|m^{(k)}(A)s| \leq |ms|(V) \leq m_p(V) \cdot p(s)$ for all $s \in E$.*
2. *$m^{(k)} \in M_p(X^{(k)}, E')$.*
3. *$m^{(k)}s \in M_\tau(X^{(k)})$ for all $s \in E$.*
4. *If $m \in M_{t,p}(X, E')$, then $m^{(k)} \in M_{t,p}(X^{(k)}, E')$.*

Proof: Let $s \in E$, $h = \chi_A s$ and $x \in A \subset V$. If W is a clopen subset of X contained in V, then $|m(W)h(x)| \leq |ms|(V)$ and so $Q_{m,h}(x) \leq |ms|(V)$, which implies that

$$|m^{(k)}(A)s| \leq \sup_{x \in A} Q_{m,h}(x) \leq |ms|(V) \leq m_p(V) \cdot p(s).$$

This proves that $m^{(k)}(A) \in E'$ and $\|m^{(k)}(A)\|_p \leq m_p(V)$. Clearly $m^{(k)} \in M_p(X^{(k)}, E')$ and $\|m^{(k)}\|_p \leq \|m\|_p$.
Let now $s \in E$ and $\epsilon > 0$. There exists a compact subset Y of X such that $|ms|(Z) < \epsilon$ for each $Z \in K(X)$ disjoint from Y. Let $B \in K_c(X)$ be disjoint from Y and let $x \in B$. Then $x \notin Y$ and so there exists a $D \in K(X)$ containing x ad contained in Y^c. For $h = \chi_B s$, we have $Q_{m,h}(x) \leq |ms|(D) < \epsilon$. Thus $|m^{(k)}(A)s| \leq \epsilon$. It follows that $|m^{(k)}s|(B) \leq \epsilon$ for each $B \in K_c(X)$ disjoint from Y and so $m^{(k)}s \in M_\tau(X^{(k)})$. Finally, assume that $m \in M_{t,p}(X.E)$. Given $\epsilon > 0$, there exists a compact subset Y of X such that $m_p(V) < \epsilon$ for each $V \in K(X)$ disjoint from Y. If $s \in E$, with $p(s) > 0$, then for $V \in K(X)$ disjoint from Y we have $|ms|(V) \leq m_p(V) \cdot p(s) < \epsilon \cdot p(s)$. Thus, for $B \in K_c(X)$ disjoint from Y we have $|m^{(k)}s|(B) \leq \epsilon \cdot p(s)$ and so $m_p^{(k)}(B) \leq \epsilon$. This clearly completes the proof.

Theorem 3.17 *Let $m \in M(X, E')$ be such that $ms \in M_\tau(X)$ for each $s \in E$. Then:*

1. *If $A \in K(X)$, then $|ms|(A) = |m^{(k)}s|(A)$ for all $s \in E$.*

2. *If $m \in M_p(X, E')$, then $m_p(A) = m_p^{(k)}(A)$ for each $A \in K(X)$.*

3. *If $f \in E^X$ is Q-integrable with respect to m, then f is Q-integrable with respect to $m^{(k)}$ and $Q_{m,f} \le Q_{m^{(k)},f}$. Moreover*

$$(Q)\int f\,dm = (Q)\int f\,dm^{(k)}.$$

Proof: Let $A \in K(X)$. Clearly $|ms|(A) \le |m^{(k)}s|(A)$. On the other hand, let $|m^{(k)}s|(A) > \theta > 0$. There exists $D \in K_c(X)$, $D \subset A$, such that $|m^{(k)}(D)s| > \theta$. Let $h = \chi_D s$. Since $|m^{(k)}(D)s| \le \sup_{x\in D} Q_{m,h}(x)$, there exists $x \in D$ such that $Q_{m,h}(x) > \theta$. The set $Y = \{z \in X : Q_{m,h}(z) \ge \theta\}$ is compact. Hence there exists $Z \in K(X)$ with $Z \cap Y = D \cap Y$. Since $x \in Z \cap A$ and $Q_{m,h}(X) > \theta$, there exists $W \in K(X)$ contained in $Z \cap A$ and such that $|m(W)h(x)| > \theta$. Then $h(x) = s$ and so $|m(W)s| > \theta$, which proves that $|ms|(A) > \theta$. Thus, $|ms|(A)| \ge |m^{(k)}s|(A)$. Assume next that $m_p^{(k)}(A) > \alpha > 0$. There exists $B \in K_c(X)$ contained in A and $s \in E$ with $|m^{(k)}(B)s|/p(s) > \alpha$. Now $|ms|(A) = |m^{(k)}s|(A) > \alpha \cdot p(s)$. Thus $m_p(A) \ge |ms|(A)/p(s) > \alpha$, which shows that $m_p(A) = m_p^{(k)}(A)$. Thus (1) and (2) hold.
(3). Assume that $f \in E^X$ is Q-integrable with respect to m.
Claim : If $x \in D \in K(X)$, then

$$\sup_{Z\in K_c(X), Z\subset D} |m^{(k)}(Z)f(x)| = \sup_{Z\in K(X), Z\subset D} |m(Z)f(x)|.$$

Indeed, suppose that there exists a $Z \in K_c(X)$ contained in D such that $|m^{(k)}(Z)f(x)| > \theta > 0$. For $h = \chi_Z f(x)$, we have

$$\theta < |m^{(k)}(Z)f(x)| \le \sup_{z\in Z} Q_{m,h}(z).$$

Thus, there exists $z \in Z$ with $Q_{m,h}(z) > \theta$. Since $z \in Z \subset D$, there exists $W \in K(X)$ contained in D such that $|m(W)h(z)| = |m(W)f(x)| > \theta$. This clearly proves the claim. Now

$$\begin{aligned} Q_{m,f}(x) &= \inf_{x\in D\in K(X)} \sup_{D\supset Z\in K(X)} |m(Z)f(x)| \\ &= \inf_{x\in D\in K(X)} \sup_{D\supset Z\in K_c(X)} |m^{(k)}(Z)f(x)| \ge Q_{m^{(k)},f}(x). \end{aligned}$$

Since f is Q-integrable with respect to m, there exists a sequence $(g_n) \subset S(X, E) \subset S(X^{(k)}, E)$ such that $\|f - g_n\|_{Q_m} \to 0$. But then $\|f - g_n\|_{Q_{m^{(k)}}} \le \|f - g_n\|_{Q_m} \to 0$. Hence f is Q-integrable with respect to $m^{(k)}$ and

$$(Q)\int f\,dm^{(k)} = \lim_{n\to\infty}\int g_n\,dm^{(k)} = \lim_{n\to\infty}\int g_n\,dm = (Q)\int f\,dm.$$

This completes the proof of the Theorem.

Next we recall the definition of the topology $\bar{\beta}_o$ which was given in [14]. Let $C_{b,k}(X,E)$ be the space of all bounded E-valued functions on X whose restriction to every compact subset of X is continuous. By Theorem 3.14 we have that $C_{b,k}(X,E) = C_b(X^{(k)},E)$. For $p \in cs(E)$, we denote by $\bar{\beta}_{o,p}$ the locally convex topology on $C_{b,k}(X,E)$ generated by the seminorms $f \to \|hf\|_p$, $h \in B_o(X)$. Since X and $X^{(k)}$ have the same compact sets, we have that $B_o(X) = B_o(X^{(k)})$ and so $\bar{\beta}_{o,p}$ coincides with the topology $\beta_{o,p}$ on $C_b(X^{(k)},E)$. The topology $\bar{\beta}_o$ is defined to be the locally convex projective limit of the topologies $\bar{\beta}_{o,p}$, $p \in cs(E)$. Thus $\bar{\beta}_o$ coincides with topology β_o on $C_b(X^{(k)},E)$.

Theorem 3.18 *1. If $m \in M_t(X,E')$, then every $f \in C_{b,k}(X,E)$ is Q-integrable with respect to m and*

$$(Q)\int f\,dm = \int f\,dm^{(k)}.$$

Thus the map

$$\phi_m : C_{b,k}(X,E) \to \mathbb{K}, \quad \phi_m(f) = (Q)\int f\,dm$$

is $\bar{\beta}_o$-continuous.

2. If E is polar, then every $\bar{\beta}_o$-continuous linear functional ϕ on $C_{b,k}(X,E)$ is of the form ϕ_m for some $m \in M_t(X,E')$.

Proof: 1. Let $p \in cs(E)$ be such that $m \in M_{t,p}(X,E')$ and $\|m\|_p < 1$. Let $d > \|f\|_p$ and $\epsilon > 0$. There exists a compact subset Y of X such that $m_p(V) < \epsilon/d$ for every $V \in K(X)$ disjoint from Y. For each $x \in Y$, the set

$$D_x = \{y \in Y : p(f(y) - f(x)) < \epsilon\}$$

is clopen in Y and $D_x = D_y$ if $D_x \cap D_y \neq \emptyset$. In view of the compactness of Y, there are $x_1, \ldots, x_n$ in Y such that the sets $D_{x_1}, \ldots, D_{x_n}$ form a partition of Y. For each k, there exists a clopen subset V_k of X such that $V_k \cap Y = D_{x_k}$. If $W_k = V_k \setminus \bigcup_{i \neq k} V_i$, then $W_k \cap Y = D_{x_k}$. Let $g = \sum_{k=1}^{n} \chi_{W_k} f(x_k)$. Then $\|f - g\|_{Q_m} \leq \epsilon$. Indeed, let $x \in X$.

Case I: $x \notin Y$. There is a clopen neighborhood V of x disjoint from Y. If $B \in K(X)$ is contained in V, then

$$|m(B)[f(x) - g(x)]| \leq p(f(x) - g(x)) \cdot m_p(V) \leq \epsilon$$

and so $Q_{m,f-g}(x) \leq \epsilon$.

Case II : $x \in Y$. There exists a k such that $x \in W_k$ and so $g(x) = f(x_k)$. If a clopen set B is contained in W_k, then

$$|m(B)[f(x) - g(x)]| = |m(B)[f(x) - f(x_k]| \leq m_p(V_k) \cdot p(f(x) - f(x_k)) \leq \epsilon,$$

and so again $Q_{m,f-g}(x) \leq \epsilon$. This proves that $\|f-g\|_{Q_m} \leq \epsilon$ and so f is Q-integrable. Now

$$\phi_m(f) = (Q)\int f\,dm = (Q)\int f\,dm^{(k)} = \int f\,dm^{(k)}.$$

Thus ϕ_m is $\bar{\beta}_o$-continuous on $C_{b,k}(X,E)$.
Finally assume that E is polar and let ϕ be a $\bar{\beta}_o$-continuous linear functional on $C_{b,k}(X,E)$. Since $\bar{\beta}_o$ induces the topology β_o on $C_b(X,E)$, there exists an $m \in M_t(X,E')$ such that

$$\phi(f) = \int f\,dm = (Q)\int f\,dm$$

for each $f \in C_b(X,E)$. Now ϕ and ϕ_m are both $\bar{\beta}_o$-continuous on $C_{b,k}(X,E)$ and they coincide on the $\bar{\beta}_o$-dense subspace $C_b(X,E)$ of $C_{b,k}(X,E)$. Thus $\phi = \phi_m$ and the proof is complete.

4 The Dual Space of $(C_b(X,E),\beta_1)$

For u a linear functional on $C_b(X,E)$, $p \in cs(E)$ and $h \in \mathbb{K}^X$, we define

$$|u|_p(h) = \sup\{|u(g)| : g \in C_b(X,E),\quad p\circ g \le |h|\}.$$

Theorem 4.1 *For a linear functional u on $C_b(X,E)$, the following are equivalent :*

1. *u is β_1-continuous.*
2. *For each sequence (V_n) of clopen sets, with $V_n \downarrow \emptyset$, there exists $p \in cs(E)$ such that $\|u\|_p < \infty$ and $\lim_{n\to\infty} |u|_p(\chi_{V_n}) = 0$.*
3. *For each sequence (h_n) in $C_b(X)$, with $h_n \downarrow 0$, there exists $p \in cs(E)$ such that $\|u\|_p < \infty$ and $\lim_{n\to\infty} |u|_p(h_n) \to 0$.*

Proof: (1) $\Rightarrow$ (2). Let $V_n \downarrow \emptyset$ and $H = \bigcap \overline{V_n}^{\beta_o X}$. Then $H \in \Omega_1$ and so u is $\beta_{H,p}$-continuous for some $p \in cs(E)$. Let $\epsilon > 0$ and $h \in C_H$ be such that

$$W_1 = \{f \in C_b(X,E) : \|hf\|_p \le 1\} \subset W = \{f : |u(f)| \le \epsilon\}.$$

It is easy to see that $\|u\|_p < \infty$. Let $M = \{x \in X : |h(x)| \ge 1\}$. There exists n_o such that $M \subset V_{n_o}^c$. Let now $n \ge n_o$ and $f \in C_b(X,E)$ with $p\circ f \le |\chi_{V_n}|$. Let $f_1 = \chi_M f,\quad f_2 = f - f_1$. If $x \in M$, then $x \in V_n^c$ and so $p(f(x)) = 0$. This implies that $f_1 \in W_1 \subset W$. Also, if $x \notin M$, then $|h(x)| \le 1$ and so $|h(x)|p(f(x)) \le 1$, which proves that $f_2 \in W_1$. Thus $f = f_1 + f_2 \in W$, which shows that $|u|_p(\chi_{V_n} \le \epsilon$.
(2) $\Rightarrow$ (3). Let $h_n \downarrow 0$. Without loss of generality, we may assume that $\|h_1\| \le 1$. Let $\lambda \in \mathbb{K},\quad 0 < |\lambda| < 1$ and set

$$V_n = \{x : |h_n(x)| \ge |\lambda|\}.$$

Then $V_n \downarrow \emptyset$. By (2), there exists $p \in cs(E)$ with $\|u\|_p < \infty$ and $|u|_p(\chi_{V_n}) \to 0$. We may choose p so that $\|u\|_p \le 1$. Choose n_o such that $|u|_p(\chi_{V_n}) < |\lambda|$ if $n \ge n_o$. Let now $n \ge n_o$. We will show that $|u|_p(h_n) \le |\lambda|$. In fact, let $f \in C_b(X,E)$ with $p\circ f \le |h_n|$, $g_1 = \chi_{V_n} f$, $g_2 = f - g_1$. If $x \in V_n$, then $p(g_1(x)) \le |h_n(x)|$ and so

$p \circ g_1 \leq |\chi_{V_n}|$, which implies that $|u(g_1)| \leq |\lambda|$. If $x \notin V_n$, then $p(g_2(x)) = p(f(x)) \leq |h_n(x)| < |\lambda|$. Hence $|u(g_2)| \leq \|u\|_p \cdot \|g_2\|_p \leq |\lambda|$, and therefore $|u(f)| \leq |\lambda|$. This proves that $|u|_p(h_n) \leq |\lambda|$.

(3) $\Rightarrow$ (2). It is trivial.

(2) $\Rightarrow$ (1). Let

$$W = \{f \in C_b(X, E) : |u(f)| \leq 1\}$$

and let $H \in \Omega_1$. There exists a decreasing sequence (V_n) of clopen subsets of X with $\bigcap \overline{V_n}^{\beta_o X} = H$. Let $p \in cs(E)$ be such that $\|u\|_p \leq 1$ and $|u|_p(\chi_{V_n}) \to 0$. Let λ be a nonzero element of $\mathbb{K}$ and choose n so that $|u|(\chi_{V_n}) < |\lambda|^{-1}$. Now

$$W_1 = \{f \in C_b(X, E) : \|f\|_p \leq |\lambda|, \ \ \|f\|_{V_n^c, p} \leq 1\} \subset W.$$

Indeed, let $f \in W_1$ and set $f_1 = \chi_{V_n} f, f_2 = f - f_1$. Since $|\lambda^{-1} f_1| \leq |\chi_{V_n}|$, we have that $|u(f_1)| \leq 1$. Also $|u(f_2)| \leq \|f_2\|_p \leq 1$, and so $|u(f)| \leq 1$, which proves that $W_1 \subset W$. By [13], Theorem 2.2, it follows that W is a $\beta_{H,p}$-neighborhood of zero. This, being true for all $H \in \Omega_1$, implies that W is a β_1-neighborhood of zero, i.e. u is β_1-continuous, which completes the proof.

Theorem 4.2 *For a set H of linear functionals on $C_b(X, E)$, the following are equivalent :*

1. *H is β_1-equicontinuous.*
2. *If (V_n) is a sequence of clopen subsets of X which decreases to the empty set, then there exists $p \in cs(E)$ such that $\sup_{u \in H} \|u\|_p < \infty$ and $|u|_p(\chi_{V_n}) \to 0$ uniformly for $u \in H$.*
3. *If (h_n) is a sequence in $C_b(X)$ with $h_n \downarrow 0$, then there exists $p \in cs(E)$ such that $\sup_{u \in H} \|u\|_p < \infty$ and $|u|_p(h_n) \to 0$ uniformly for $u \in H$.*

Proof: (1) $\Rightarrow$ (2). Let $V_n \downarrow \emptyset$. Then $Z = \bigcap \overline{V_n}^{\beta_o X} \in \Omega_1$. Let $\lambda \in \mathbb{K}, \lambda \neq 0$. Since H is β_1-equicontinuous, the set λH^o is a β_1-neighborhood of zero. Thus, there exists $p \in cs(E)$ such that λH^o is a $\beta_{Z,p}$-neighborhood of zero. Let $h \in C_Z$ be such that

$$W_1 = \{f : \|hf\|_p \leq 1\} \subset \lambda H^o.$$

It follows now easily that $\sup_{u \in H} \|u\|_p < \infty$. Also, as in the proof of the implication (1) $\Rightarrow$ (2) in the preceding Theorem, we prove that $|u|_p(\chi_{V_n}) \to 0$ uniformly for $u \in H$. For the proofs of the implications (2) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1) we use an argument analogous to the one used in the proof of the preceding Theorem.

Theorem 4.3 *In the space $C_b(X)$, β_1 is the finest of all locally solid topologies γ with the following property: If $(f_n) \subset C_b(X)$ with $f_n \downarrow 0$, then $f_n \xrightarrow{\gamma} 0$.*

Proof: By [12], Theorems 3.7 and 3.8, β_1 is locally solid and $f_n \xrightarrow{\beta_1} 0$ when $f_n \downarrow 0$. Consider now the family $\mathcal{U}$ of all solid absolutely convex subsets W of $C_b(X)$ such that $f_n \in W$ eventually when $f_n \downarrow 0$. Clearly $\mathcal{U}$ is a base at zero for the finest locally solid topology γ_o on $C_b(X)$ having the property mentioned in the Theorem.

Claim I : γ_o is coarser than τ_u. Indeed, let $W \in \mathcal{U}$ and let $\lambda \in \mathbb{K}$, $0 < |\lambda| < 1$. For each n, let g_n be the constant function λ^n. Since $g_n \downarrow 0$, there exists an n with $g_n \in W$. If now $f \in C_b(X)$ with $\|f\| \leq \|\lambda|^n$, then $f \in W$, which implies that W is a τ_u-neighborhood of zero.
Claim II : β_1 is finer than γ_o and hence $\beta_1 = \gamma_o$. Indeed, let $W \in \mathcal{U}$, $Z \in \Omega_1$ and $r > 0$. There exists $\epsilon > 0$ such that

$$W_1 = \{f \in C_b(X) : \|g\| \leq \epsilon\} \subset W.$$

Choose $\mu \in \mathbb{K}$ with $|\mu| \geq r$. There exists a decreasing sequence (V_n) of clopen subsets of X with $Z = \bigcap \overline{V_n}^{\beta_o X}$. Since $\mu\chi_{V_n} \downarrow 0$, there exists n such that $\mu\chi_{V_n} \in W$. Let now $f \in C_b(X)$ with $\|f\| \leq r$, $\|f\|_{V_n^c} \leq \epsilon$, and let $g = f \cdot \chi_{V_n}$, $h = f - g$. Then $|g| \leq |\mu\chi_{V_n}|$and so $g \in W$ since W is solid. Also, $\|h\| \leq \epsilon$ and so $h \in W$, which implies that $f \in W$. This proves that W is a β_Z-neighborhood of zero for all $Z \in \Omega_1$ and hence W is a β_1-neighborhood of zero. This clearly completes the proof.

The proofs of the following two Theorems are analogous to the ones of Theorems 4.2 and 4.3.

Theorem 4.4 *For a subset H of linear functionals on $C_b(X)$, the following are equivalent :*

1. *H is β-equicontinuous.*
2. *For each net (V_δ), of clopen subsets of X with $V_\delta \downarrow 0$, there exists $p \in cs(E)$ such that $\sup_{u \in H} \|u\|)p < \infty$ and $|u|_p(\chi_{V_\delta}) \to 0$ uniformly for $u \in H$.*
3. *For each net (h_δ) in $C_b(X)$ with $h_\delta \downarrow 0$, there exists $p \in cs(E)$ such that $\sup_{u \in H} \|u\|_p < \infty$ and $|u|_p(h_\delta) \to 0$ uniformly for $u \in H$.*

Theorem 4.5 *In the space $C_b(X)$, β is the finest of all locally solid topologies γ with the following property: If $(f_\delta) \subset C_b(X)$ with $f_\delta \downarrow 0$, then $f_\delta \xrightarrow{\gamma} 0$.*

5 θ_o-Complete Spaces

Recall that $\theta_o X$ is the set of all $z \in \beta_o X$ with the following property: For each clopen partition (V_i) of X there exists i such that $z \in \overline{V_i}^{\beta_o X}$ (see [1]). By [1], Lemma 4.1, we have $X \subset \theta_o X \subset \upsilon_o X$. For each clopen partition $\alpha = (V_i)_{i \in I}$ of X, let

$$W_\alpha = \bigcup_{i \in I} V_i \times V_i.$$

Then the family of all W_α, α a clopen partition of X, is a base for a uniformity $\mathcal{U}_c = \mathcal{U}_c^X$, compatible with the topology of X, and $(\theta_o X, \mathcal{U}_c^{\theta_o X})$ coincides with the completion of $(X, \mathcal{U}_c)$. We will say that X is θ_o-complete iff $X = \theta_o X$. As it is shown in [1], if Y is a θ_o-complete and $f : X \to Y$ is a continuous function, then f has a continuous extension $f^{\theta_o} : \theta_o X \to Y$. A subset A of X is called bounding if every

$f \in C(X)$ is bounded on A. Note that several authors use the term bounded set instead of bounding. But in this paper we will use the term bounding to distinguish from the notion of a bounded set in a topological vector space. A set $A \subset X$ is bounding iff $\overline{A}^{v_o X}$ is compact. In this case (as it is shown in [1], Theorem 4.6) we have that $\overline{A}^{\theta_o X} = \overline{A}^{v_o X} = \overline{A}^{\beta_o X}$. Clearly a continuous image of a bounding set is bounding. Let us say that a family $\mathcal{F}$ of subsets of X is finite on a subset A of X if the family $\{f \in \mathcal{F} : F \cap A \neq \emptyset\}$ is finite. We have the following easily established

Lemma 5.1 *For a subset A of X, the following are equivalent :*

1. *A is bounding.*
2. *Every continuous real-valued function on X is bounded on A.*
3. *Every locally finite family of open subsets of X is finite on A.*
4. *Every locally finite family of clopen subsets of X is finite on A.*

By [3], Theorem 4.6, every ultraparacompact space (and hence every ultrametrizable space) is θ_o-complete.

Theorem 5.2 *Every complete Hausdorff locally convex space E is θ_o-complete.*

Proof: Let $\mathcal{U}$ be the usual uniformity on E, i.e. the uniformity having as a base the family of all sets of the form

$$W_{p,\epsilon} = \{(x,y) : p(x-y) \leq \epsilon\},\ p \in cs(E),\ \epsilon > 0.$$

Given $W_{p,\epsilon}$, we consider the clopen partition $\alpha = (V_i)_{i\in I}$ of E generated by the equivalence relation $x \sim y$ iff $p(x-y) \leq \epsilon$. Then $W_{p,\epsilon} = W_\alpha$ and so $\mathcal{U}$ is coarser that $\mathcal{U}_c$. Since $(E,\mathcal{U})$ is complete and $\mathcal{U}_c$ is compatible with the topology of E, it follows that $(E,\mathcal{U}_c)$ is complete and the result follows.

Corollary 5.3 *A subset B, of a complete Hausdorff locally convex space E, is bounding iff it is totally bounded.*

Proof: If B is bounding, then $\overline{B} = \overline{B}^{\theta_o E}$ is compact and hence totally bounded, which implies that B is totally bounded. Conversely, if B is totally bounded, then $\overline{B}$ is totally bounded. Thus $\overline{B}$ is compact and hence B is bounding.

Theorem 5.4 *If G is a locally convex space (not necessarily Hausdorff), then every bounding subset A of G is totally bounded.*

Proof: Assume first that G is Hausdorff. Let $\hat{G}$ be the completion of G. The closure B of A in $\hat{G}$ is bounding and hence B is totally bounded, which implies that A is totally bounded. If G is not Hausdorff, we consider the quotient space $F = G/\overline{\{0\}}$ and let $u : G \to F$ be the quotient map. Since u is continuous, the set $u(A)$ is bounding, and hence totally bounded, in F. Let now V be a convex neighborhood of zero in G. Then, $u(V)$ is a neighborhood of zero in F. Let S be finite subset of A such that $u(A) \subset u(S) + u(V)$. But then

$$A \subset S + V + \overline{\{0\}} \subset S + V + V = S + V,$$

which proves that A is totally bounded.

Theorem 5.5 *1. Closed subspaces of θ_o-complete spaces are θ_o-complete.*

2. *If $X = \prod X_i$, with $X_i \neq \emptyset$ for all i, then X is θ_o-complete iff each X_i is θ_o-complete .*

3. *If $(Y_i)_{i\in I}$ is a family of θ_o-complete subspaces of X, then $Y = \bigcap Y_i$ is θ_o-complete.*

4. *$\theta_o X$ is the smallest of all θ_o-complete subspaces of $\beta_o X$ which contain X.*

Proof: (1). Let Z be a closed subspace of a θ_o-complete space X and let (x_δ) be a $\mathcal{U}_c^Z$-Cauchy net in Z. Then (x_δ) is $\mathcal{U}_c^X$-Cauchy and hence $x_\delta \to x \in X$. Moreover, $x \in Z$ since Z is closed.
(2). Each X_i is homeomorphic to a closed subspace of X. Thus X_i is θ_o-complete if X is θ_o-complete. Conversely, suppose that each X_i is θ_o-complete. If (x^δ) is a $\mathcal{U}_c^X$-Cauchy net, then (x_i^δ) is a $\mathcal{U}_c^{X_i}$-Cauchy net in X_i and hence $x_i^\delta \to x_i \in X_i$. If $x = (x_i)$, then $x^\delta \to x$, which proves that $(X, \mathcal{U}_c)$ is complete.
(3). Let $X = \prod Y_i$ and consider the map $f : Y \to X$, $f(x)_i = x$ for all i. Then $f : Y \to f(Y) = D$ is a homeomorphism. Also D is a closed subspace of X. Since X is θ_o-complete, it follows that D is θ_o-complete and hence Y is θ_o-complete.
(4). Since $\theta_o X$ is θ_o-complete (by [1], Theorem 4,9) and $X \subset \theta_o X \subset \beta_o X$, the result follows from (3).

Theorem 5.6 *For a point $z \in \beta_o X$, the following are equivalent :*

1. *$z \in \theta_o X$.*

2. *If Y is a Hausdorff ultraparacompact space and $f : X \to Y$ continuous, then $f^{\beta_o}(z) \in Y$, where $f^{\beta_o} : \beta_o X \to \beta_o Y$ is the continuous extension of f.*

3. *For every ultrametric space Y and every $f : X \to Y$ continuous , we have that $f^{\beta_o}(z) \in Y$.*

Proof: (1) $\Rightarrow$ (2). Since $\theta_o Y = Y$, the result follows from [1], Theorem 4.4.
(2) $\Rightarrow$ (3). It is trivial.
(3) $\Rightarrow$ (1). Assume that $z \notin \theta_o X$. Then, there exists a clopen partition (A_i) of X such that $z \notin \bigcup_i \overline{A_i}^{\beta_o X}$. Let $f_i = \chi_{A_i}$ and define

$$d : X \times X \to \mathbf{R}, \quad d(x,y) = \sup_i |f_i(x) - f_i(y)|.$$

Then d is a continuous ultrapseudometric on X. Let $Y = X_d$ be the corresponding ultrametric space and let $\pi : X \to Y_d$ be the quotient map, $x \mapsto \tilde{x}_d = \tilde{x}$. Since π is continuous, there exists (by (3)) an $x \in X$ such that $\pi^{\beta_o}(z) = \tilde{x}_d$. Let (x_δ) be a net in X converging to z. Then $\tilde{x_\delta} = \pi^{\beta_o}(x_\delta) \to \pi^{\beta_o}(z) = \tilde{x}$, and so $d(x_\delta, x) \to 0$. If $x \in A_i$, then $|f_i(x_\delta) - 1| \to 0$, and so there exists δ_o such that $x_\delta \in A_i$ when $\delta \geq \delta_o$. But then $z \in \overline{A_i}^{\beta_o X}$, a contradiction. This completes the proof.

Theorem 5.7 *Let X be a dense subspace of a Hausdorff zero-dimensional space Y. The following are equivalent :*

1. *$Y \subset \theta_o X$ (more precisely, Y is homeomorphic to a subspace of $\theta_o X$).*
2. *Each continuous function, from X to any ultrametric space Z, has a continuous extension to all of Y.*

Proof: (1) implies (2) by the preceding Theorem.
(2) $\Rightarrow$ (1). We will prove first that, for each clopen subset V of X, we have that $\overline{V}^Y \cap \overline{V^c}^Y = \emptyset$, and so $\overline{V}^Y$ is clopen in Y. Indeed, define

$$d : X \times X \to \mathbf{R}, \quad d(x,y) = \max\{|f_1(x) - f_1(y)|, |f_2(x) - f_2(y)|\},$$

where $f_1 = \chi_V, f_2 = \chi_{V^c}$. Then d is a continuous ultrapseudometric on X. Let $\pi : X \to X_d$ be the quotient map. By our hypothesis, there exists a continuous extension $h : Y \to X_d$ of π. Suppose that $z \in \overline{V}^Y \cap \overline{V^c}^Y$. There are nets $(x_\delta), (y_\gamma)$, in V, V^c respectively, such that $x_\delta \to z$, and $y_\gamma \to z$. Let $\tilde{d}$ be the ultrametric of X_d and let δ_o, γ_o be such that

$$\tilde{d}(\pi(x_\delta), h(z)) < 1 \quad \textit{and} \quad \tilde{d}(\pi(x_\gamma), h(z)) < 1$$

when $\delta \geq \delta_o, \gamma \geq \gamma_o$. Now

$$d(x_{\delta_o}, y_{\gamma_o}) = \tilde{d}(\pi(x_{\delta_o}), \pi(y_{\delta_o})) < 1,$$

a contradiction. Thus $\overline{V}^Y$ is clopen in Y. If $A = \overline{V}^Y, B = \overline{V^c}^Y$, then

$$\overline{A}^{\beta_o Y} \bigcap \overline{B}^{\beta_o Y} = \overline{V}^{\beta_o Y} \bigcap \overline{V^c}^{\beta_o Y} = \emptyset.$$

This, being true for each clopen subset V of X, implies that $\beta_o X = \beta_o Y$ and so $X \subset Y \subset \beta_o Y = \beta_o X$. Now our hypothesis (2) and the preceding Theorem imply that $Y \subset \theta_o X$, and the result follows.

Theorem 5.8 *For each continuous ultrapseudometric d on X, there exists a continuous ultrapseudometric d^{θ_o} on $\theta_o X$ which is an extension of d. Moreover, d^{θ_o} is the unique continuous extension of d.*

Proof: Consider the ultrametric space X_d and let $\tilde{d}$ be its ultrametric. Let h be the coninuous extension of the quotient map $\pi : X \to X_d$ to all of $\theta_o X$. Define

$$d^{\theta_o} : \theta_o X \times \theta_o X \to \mathbf{R},\ d^{\theta_o}(y,z) = \tilde{d}(h(y), h(z)).$$

It is easy to see that d^{θ_o} is a continuous ultrapseudometric which is an extension of d. Finally, let ϱ be any continuous ultrapseudometric on $\theta_o X$, which is an extension of d, and let $y, z \in \theta_o X$. There are nets $(y_\delta)_{\delta\in\Delta}$, $(z_\gamma)_{\gamma\in\Gamma})$ in X which convergence to y, z, respectively. Let $\Phi = \Delta \times \Gamma$ and consider on Φ the order $(\delta_1, \gamma_1) \geq (\delta_2, \gamma_2)$ iff $\delta_1 \geq \delta_2$ and $\gamma_1 \geq \gamma_2$. For $\phi = (\delta, \gamma) \in \Phi$, we let $a_\phi = y_\delta$, $b_\phi = z_\gamma$. Then $a_\phi \to y$, $b_\phi \to z$. Thus

$$\begin{aligned} \varrho(y,z) &= \lim \varrho(a_\phi, b_\phi) = \lim \tilde{d}(h(a_\phi), h(b_\phi)) && (1) \\ &= \lim d^{\theta_o}(a_\phi, b_\phi) = d^{\theta_o}(y,z) && (2) \end{aligned}$$

and hence $\varrho = d^{\theta_o}$, which completes the proof.

Theorem 5.9 *Let (H_n) be a sequence of equicontinuous subsets of $C(X)$. If $z \in \theta_o X$, then there exists $x \in X$ such that $f^{\theta_o}(z) = f(x)$ for all $f \in \bigcup H_n = H$.*

Proof: Define

$$d : X^2 \to \mathbf{R}, \quad d(x,y) = \max_n \min\{1/n, \sup_{f\in H_n} |f(x) - f(y)|\}.$$

Then d is a continuous ultrapseudometric on X. Take $Y = X_d$ and let $\pi : X \to Y$ be the quotient map. Then $\pi^{\beta_o}(z) = u \in Y$. Choose $x \in X$ with $\pi(x) = u$, and let (x_δ) be a net in X converging to z in $\beta_o X$. Now $f(x_\delta) \to f^{\beta_o}(z)$ for all $f \in H$. Since $\pi(x_\delta) \to \pi(x)$, we have that $d(x_\delta, x) \to 0$, and so $|f(x_\delta) - f(x)| \to 0$ for all $f \in H$. Thus, for $f \in H$, we have $f(x) = \lim f(x_\delta) = f^{\beta_o}(z)$, and the result follows.

Theorem 5.10 *If $H \subset C(X)$ is equicontinuous, then the family*

$$H^{\theta_o} = \{f^{\theta_o} : f \in H\}$$

is equicontinuous on $\theta_o X$. Moreover, if H is pointwise bounded, then the same holds for H^{θ_o}

Proof: Define

$$d : X^2 \to \mathbf{R}, \quad d(x,y) = \min\{1, \sup_{f\in H} |f(x) - f(y)|\}.$$

Let $\pi^{\theta_o} : \theta_o X \to X_d$ be the continuous extension of the quotient map $\pi : X \to X_d$. Let $z \in \theta_o X$ and $\epsilon > 0$. There exists $x \in X$ such that $\pi^{\theta_o}(z) = \pi(x)$. Let (x_δ) be a net in X converging to z. Then $\pi(x_\delta) \to \pi^{\theta_o}(z) = \pi(x)$ and so $d(x_\delta, x) \to 0$. Thus, for $f \in H$, we have $f^{\theta_o}(z) = \lim f(x_\delta) = f(x)$. The set $W = \{y \in X : d(x,y) \leq \epsilon\}$ is d-clopen (hence clopen) in X and so $\overline{W}^{\theta_o X} = V$ is clopen in $\theta_o X$. Since $x_\delta \in W$ eventually, it follows that $z \in V$. Now, for $f \in H$ and $a \in V$, we have that $|f^{\theta_o}(a) - f^{\theta_o}(z)| \leq \epsilon$. In fact, there exists a net (y_γ) in W converging to a. Thus

$$|f^{\theta_o}(a) - f^{\theta_o}(z)| = |f(x) - f^{\theta_o}(a)| = \lim_\gamma |f(x) - f(y_\gamma)| \leq \epsilon.$$

This proves that H^{θ_o} is equicontinuous on $\theta_o X$. The last assertion follows from the preceding Theorem.

Theorem 5.11 *$\mathcal{U}_c = \mathcal{U}_c^X$ is the uniformity $\mathcal{U}$ generated by the family of all continuous ultrapseudometrics on X.*

Proof: Let (A_i) be a clopen partition of X and let $W = \bigcup A_i \times A_i$. Define

$$d(x, y) = \sup_i |f_i(x) - f_i(y)|,$$

where $f_i = \chi_{A_i}$. Then d is a continuous ultrapseudometric on X. Since

$$W = \{(x, y) : d(x, y) < 1/2\},$$

it follows that $\mathcal{U}_c$ is coarser than $\mathcal{U}$. Conversely, let d be a continuous ultrapseudometric on X, $\epsilon > 0$ and $D = \{(x, y) : d(x, y) \leq \epsilon\}$. If α is the clopen partition of X corresponding to the equivalence relation $x \sim y$ iff $d(x, y) \leq \epsilon$, then $D = W_\alpha$ and the result follows.

Theorem 5.12 *Let $(Y_i, f_i)_{i \in I}$ be the family of all pairs (Y, f), where Y is an ultrametric space and $f : X \to Y$ a continuous map. Then*

$$\theta_o X = \bigcap_{i \in I} (f_i^{\beta_o})^{-1}(Y_i).$$

Proof: It follows from Theorem 5.6.

Theorem 5.13 *A Hausdorff zero-dimensional space X is θ_o-complete iff it is homeomorphic to a closed subspace of a product of ultrametic spaces.*

Proof: Every ultrametric space is θ_o-complete. Thus the sufficiency follows from Theorem 5.5. Conversely, assume that X is θ_o-complete and let $(Y_i, f_i)_{i \in I}$ be as in the preceding Theorem. Then $X = \bigcap_i Z_i$, $Z_i = (f_i^{\beta_o})^{-1}(Y_i)$. Let $Y = \prod Y_i$ with its product topology. The map $u : X \to Y$, $u(x)_i = f_i(x)$, is one-to-one. Indeed, let $x \neq y$ and choose a clopen neighborhood V of x not containing y. Let $f = \chi_V$ and

$$d : X \times Y \to \mathbf{R}, \quad d(a, b) = |f(a) - f(b)|.$$

The quotient map $\pi : X \to X_d$ is continuous and $\pi(x) \neq \pi(y)$, which implies that $u(x) \neq u(y)$. Clearly u is continuous. Also $u^{-1} : u(X) \to X$ is continuous. Indeed, let V be a clopen subset of X containing x_o and consider the pseudometric $d(x, y) = |\chi_V(x) - \chi_V(y)|$. Let $\pi : X \to X_d$ be the quotient map. There exists a $i \in I$ such that $Y_i = X_d$ and $f_i = \pi$. Then

$$f_i(V) = \pi(V) = \{\pi(x) : \tilde{d}(\pi(x) - \pi(x_o)) < 1\}.$$

The set $\pi(V)$ is open in $Y_i = X_d$. Let $\pi_i : Y \to Y_i$ be the *ith*-projection map and $G = \pi_i^{-1}(\pi(V))$. If $x \in X$ is such that $u(x) \in G$, then $f_i(x) = u(x)_i \in \pi(V)$ and so $d(x, x_o) < 1$, which implies that $x \in V$ since $x_o \in V$. This proves that $u : X \to u(X)$ is a homeomorphism. Finally, $u(X)$ is a closed subspace of Y. In fact, let (x_δ) be a net in X with $u(x_\delta) \to y \in Y$. Then $f_i(x_\delta) \to y_i$ for all i. Going to a subnet if necessary, we may assume that $x_\delta \to z \in \beta_o X$. Now $f_i(x_\delta) \to f_i^{\beta_o}(z)$ in $\beta_o Y_i$. But then $f_i^{\beta_o}(z) = y_i \in Y_i$, for all i, and hence $z \in \theta_o X = X$, by Theorem 5.12. Thus $y_i = f_i(z)$, for all i, and hence $y = u(z)$. This proves that X is homeomorphic to a closed subspace of Y and the result follows.

Corollary 5.14 *Every Hausdorff ultraparacompact space is homeomorphic to a closed subspace of a product of ultrametric spaces.*

Theorem 5.15 *For a subset A of X, the following are equivalent :*

1. *A is bounding.*
2. *A is $\mathcal{U}_c$-totally bounded.*
3. *For each continuous ultrapseudometric d on X, A is d-totally bounded.*

Proof: In view of Theorem 5.11, (2) is equivalent to (3). Also, by [1], Theorem 4.11, (1) implies (2).
(2) ⇒ (1). Let $f \in C(X)$,

$$A_1 = \{x : |f(x)| \leq 1\}, \quad A_{n+1} = \{x : n < |f(x)| \leq n+1\}$$

for $n \geq 1$. Then (A_n) is a clopen partition of X. Let $W = \bigcup_n A_n \times A_n$. By our hypothesis, there are $x_1, \ldots, x_N$ in A such that $A \subset \bigcup_1^N W[x_k]$. For each $1 \leq k \leq N$, there exists n_k such that $x_k \in A_{n_k}$. Then $A \subset \bigcup_1^N A_{n_k}$ and so

$$\|f\|_A \leq \max_{1 \leq k \leq N} n_k,$$

which proves that A is bounding.

Theorem 5.16 *Let X, Y be zero-dimensional Hausdoff spaces. If one of the spaces X, Y is finite, then $\beta_o(X \times Y) = \beta o X \times \beta_o Y$.*

Proof: Assume (say) that $X = \{x_1 \ldots, x_n\}$ and let Z be a zero-dimensiomnal Hausdorff compact space and $h : X \times Y \to Z$ continuous. We will prove that there exists a continuous extension $\hat{h} : X \times \beta_o Y \to Z$. Indeed, for $1 \leq k \leq n$, let $g_k : Y \to Z$, $g_k(y) = h(x_k, y)$. Since g_k is continuous, there exists a continuous extension $g_k^{\beta_o} : \beta_o Y \to Z$ of g_k. Now it suffices to take

$$\hat{h} : X \times \beta_o Y \to Z, \quad \hat{h}(x, y) = g_k^{\beta_o}(y) \quad if \quad x = x_k.$$

It follows from this that $X \times \beta_o Y$ is homeomorphic to $\beta_o(X \times Y)$.

As the following Theorem shows, it is not in general true that $\beta_o(X \times Y) = \beta_o X \times \beta_o Y$.

Theorem 5.17 *Let X, Y be zero-dimensional Hausdorff topological spaces such that $\beta_o(X \times Y) = \beta o X \times \beta_o Y$. If both X, Y are infinite, then X, Y, $X \times Y$ are pseudo-compact.*

Proof: We prove first that each of the spaces X, Y is pseudocompact. To show that X is pseudocopact, we consider the two possible cases for Y.
Case I. Y is compact. Suppose that X is not pseudocompact and let $z \in \beta_o X \backslash \upsilon_o X$. There exists a decreasing sequence (Z_n) of clopen neighborhoods of z in $\beta_o X$ with $\bigcap V_n = \emptyset$, where $V_n = Z_n \bigcap X$. Without loss of generality, we may assume that (V_n)

is strictly decreasing. Let $A_o = \emptyset$ and $A_n = X \setminus V_n$ for $n \in \mathbf{N}$. Let $\lambda \in \mathbb{K}$, $0 < |\lambda| < 1$, and define

$$g : X \to \mathbb{K}, \quad g(x) = \lambda^k \quad if \quad x \in A_k \setminus A_{k-1}.$$

Then g is continuous and $g(x) \neq 0$ for each $x \in X$. Choose $x_n \in A_n \setminus A_{n-1}$, $n \in \mathbf{N}$. Since Y is infinite, there exists (by [15], Lemma 2.5) an infinite sequence (B_n) of pairwise disjoint clopen subsets of Y. Let $y_n \in B_n$ and take $f = \sum_{n=1}^{\infty} \lambda^n \chi_{B_n}$. Then f is continuous. Let

$$D_1 = \{\alpha \in \mathbb{K} : |\alpha| \geq |\lambda|\}, \quad D_{n+1} = \{\alpha \in \mathbb{K} : |\lambda|^{n+1} \leq |\alpha| < |\lambda|^n\}.$$

Define $h : \mathbb{K} \to \mathbb{K}$, $h = \sum_{n=1}^{\infty} \lambda^n \chi_{D_n}$. Clearly h is continuous. Let

$$u : X \times Y \to \mathbb{K}, \quad u(x, y) = \frac{h(f(y))}{g(x)}.$$

Let $(x, y) \in \beta_o X \times Y$ be a cluster point of the sequence $((x_n, y_n))_{n \in \mathbf{N}}$. There exists a subnet $((x_{\phi(\gamma)}, y_{\phi(\gamma)}))_{\gamma \in \Gamma}$ of the sequence $((x_n, y_n))_{n \in \mathbf{N}}$ converging to (x, y). For each n we have $u(x_n, y_n) = 1$. Since

$$X \times Y \subset \beta_o X \times Y = \beta_o(X \times Y),$$

we have that

$$u^{\beta_o}(x, y) = \lim_{\gamma} u(x_{\phi(\gamma}, y_{\phi(\gamma)}) = 1.$$

Since $f(y_n) = \lambda^n$ and $(f(y_{\phi(\gamma)}))_{\gamma \in \Gamma}$ is a subnet of the sequence $(f(y_n))$, it follows that $f(y) = \lim_{\gamma} f(y_{\phi(\gamma)}) = 0$ and so $u(z, y) = \frac{h(f(y))}{g(z)} = 0$, for all $z \in X$. Thus

$$u^{\beta_o}(x, y) = \lim_{\gamma} u^{\beta_o}(x_{\phi(\gamma)}, y) = \lim_{\gamma} u(x_{\phi(\gamma}, y) = 0,$$

a contradiction.

Case II. Y not necessarily compact. We have that

$$X \times Y \subset X \times \beta_o Y \subset \beta_o X \times \beta_o Y = \beta_o(X \times Y).$$

Thus $\beta_o(X \times \beta_o Y) = \beta_o X \times \beta_o Y$. By case I, X is pseudocompact. Finally, assume that $X \times Y$ is not pseudocompact and let

$$(x_o, y_o) \in \beta_o(X \times Y) \setminus v_o(X \times Y).$$

Let (W_n) be a decreasing sequence of clopen neighborhoods of (x_o, y_o) such that $\bigcap W_n \bigcap (X \times Y) = \emptyset$. Let $0 < |\lambda| < 1$, $V_1 = W_1^c$, $V_{n+1} = W_n \setminus W_{n+1}$ and $f = \sum_{n=1}^{\infty} \lambda^n \chi_{V_n}$. Then f is continuous on $\beta_o(X \times Y)$ and $f(x, y) \neq 0$ for each $(x, y) \in X \times Y$. Let $x \in X$. The function $h : \beta_o Y \to \mathbb{K}$, $h(y) = f(x, y)$, is continuous and $h(y) \neq 0$ for each $y \in Y$. Since Y is pseudocompact, it follows that $h(y) \neq 0$ for each $y \in \beta_o Y$. Now consider the function

$$\phi : \beta_o X \to \mathbb{K}, \quad \phi(z) = f(z, y_o).$$

Then $\phi(z) \neq 0$ for $z \in X$, and so $\phi(z) \neq 0$ for all $z \in \beta_o X$ since X is pseudocompact. In particular $f(x_o, y_o) = \phi(x_o) \neq 0$, a contradiction. This completes the proof.

Theorem 5.18 *Let X, Y be zero-dimemnsional Hausdorff topological spaces such that $\beta_o(X \times Y) = \beta_o X \times \beta_o Y$. Then $\theta_o(X \times Y) = \theta_o X \times \theta_o Y$.*

Proof: The space $Z = \theta_o X \times \theta_o Y$ is θ_o-complete and $X \times Y \subset Z \subset \beta_o(X \times Y)$. Hence $\theta_o(X \times Y) \subset Z$. Let G be an ultrametric space, $f : X \times Y \to G$ a continuous function, and $z \in \theta_o Y$. Consider the function

$$g : X \to G, \quad g(x) = f^{\beta_o}(x, z),$$

where $f^{\beta_o} : \beta_o(X \times Y) \to \beta_o G$ is the continuous extension of f. Since the map $a \mapsto f^{\beta_o}(a, z)$ is continuous on $\beta_o X$, it follows that $g^{\beta_o}(a) = f^{\beta_o}(a, z)$ for all $a \in \beta_o X$. For $a \in \theta_o X$, we have $g^{\beta_o}(a) \in G$. So, for $(a, z) \in \theta_o X \times \theta_o Y$, we have that $f^{\beta_o}(a, z) \in G$. This (by Theorem 4.6) implies that $Z \subset \theta_o(X \times Y)$ and so $Z = \theta_o(X \times Y)$. Hence the result follows.

Theorem 5.19 *For an $m \in M(X)$, the following are equivalent:*

1. *$supp(m^{\beta_o}) \subset \theta_o X$.*
2. *If $(V_i)_{i \in I}$ is a clopen partition of X, then there exists a finite subset J of I such that $|m|(\bigcup_{i \notin J} V_i) = 0$.*
3. *If (V_δ) is a net of clopen subsets of X such that $\overline{V_\delta}^{\beta_o X} \downarrow H \in \Omega_u$, then there exists a δ_o such that $m(V_\delta) = 0$ for $\delta \geq \delta_o$.*
4. *If $\overline{V_\delta}^{\beta_o X} \downarrow H \in \Omega_u$, then there exists a δ such that $|m|(V_\delta) = 0$.*
5. *For each $H \in \Omega_u$, there exists a clopen subset A of X with $|m|(A) = 0$ and $H \subset \overline{A}^{\beta_o X}$.*

(1) $\Rightarrow$ (2). Since

$$supp(m^{\beta_o}) \subset \theta_o X \subset \bigcup \overline{V_i}^{\beta_o X},$$

there exists a finite set J such that

$$supp(m^{\beta_o}) \subset \theta_o X \subset \bigcup_{i \in J} \overline{V_i}^{\beta_o X}.$$

If now $W \in K(X)$ is contained in $\bigcup_{i \notin J} V_i$, then $\overline{W}^{\beta_o X}$ is disjoint from supp(m^{β_o}) and so $m(W) = 0$. It follows that $|m|(\bigcup_{i \notin J} V_i) = 0$.
(2) $\Rightarrow$ (3). Since $H \in \Omega_u$, there exists a clopen partition $(A_i)_{i \in I}$ of X such that H is disjoint from each $\overline{A_i}^{\beta_o X}$. By our hypothesis, there exists a finite subset J of I such that $|m|(A) = 0$, where $A = \bigcup_{i \notin J} A_i$. If $B = X \setminus A$, then $H \bigcap \overline{B}^{\beta_o X} = \emptyset$ and so $H \subset \overline{A}^{\beta_o X}$. Now $\overline{B}^{\beta_o X} \subset \bigcup_\delta \overline{V_\delta^c}^{\beta_o X}$, and so $\overline{B}^{\beta_o X} \subset \overline{V_{\delta_o}^c}^{\beta_o X}$, for some δ_o. If $\delta \geq \delta_o$, then $V_\delta \subset A$ and so $m(V_\delta) = 0$.
The proofs of the implications (3) $\Rightarrow$ (4) $\Rightarrow$ (5) are analogous to the ones used in Theorem 2.5.
(5) $\Rightarrow$ (1). Assume that there exists $z \in supp(m^{\beta_o})$, $z \notin \theta_o X$. Then there exists a clopen partition $(A_i)_{i \in I}$ of X with $z \notin \bigcup \overline{A_i}^{\beta_o X}$. Thus $\{z\} \in \Omega_u$. By (5), there exists a clopen subset A of X with $z \in \overline{A}^{\beta_o X} = D$ and $|m|(A) = 0$. But then $|m^{\beta_o}|(D) = |m|(A) = 0$, contradicting the fact that z is in the support of m^{β_o}. This completes the proof.

6 The Space $M_b(X, E')$

We denote by $M_b(X,E')$ the space of all $m \in M(X,E')$ which have a bounding support, i.e. there exists a bounding subset B of X such that $m(V) = 0$ for all clopen V disjoint from B. In case $E = \mathbb{K}$, we write simply $M_b(X)$.

Theorem 6.1 *If $m \in M_b(X,E')$, then every $f \in C(X,E)$ is m-integrable. Moreover, if B is a bounding support of m and $p \in cs(E)$ with $m_p(X) < \infty$, then*

$$\left|\int f\,dm\right| \leq \|f\|_{B,p} \cdot \|m\|_p.$$

Proof: Let $f \in C_b(X,E)$ and let B be a bounding subset of X which is a support set for m. Since the the closure of a bounding set is bounding, we may assume that B is closed. Let $p \in cs(E)$ with $m_p(X) < \infty$. The set $f(B)$ is bounding in E and hence totaly bounded by Theorem 5.4. Thus, given $\epsilon > 0$, there are $x_1, \ldots, x_n$ in B such that the sets

$$V_k = \{x : p(f(x) - f(x_k)) \leq \epsilon/\|m\|_p\}, \quad k = 1, \ldots, n,$$

are pairwise disjoint and cover B. Let $V_{n+1} = X \setminus \bigcup_{k=1}^n V_k$ and choose $x_{n+1} \in V_{n+1}$ if $V_{n+1} \neq \emptyset$. Let $\{W_1, \ldots, W_N\}$ be a clopen partition of X which is a refinement of $\{V_1, \ldots, V_{n+1}\}$ and $y_j \in W_j$. We may assume that $\bigcup_{i=1}^n V_i = \bigcup_{j=1}^k W_j$. If $W_j \subset V_i$ for some $i \leq n$, then

$$|m(W_j)[f(y_j) - f(x_i)]| \leq \|m\|_p \cdot p(f(y_j) - f(x_i)) \leq \epsilon,$$

while, for $W_j \subset V_{n+1}$, we have $m(W_j) = 0$. Thus

$$\left|\sum_{j=1}^N m(W_j)f(y_j) - \sum_{i=1}^n m(V_i)f(x_i)\right| \leq \epsilon.$$

This proves that f is m-integrable and

$$\left|\int f\,dm - \sum_{i=1}^n m(V_i)f(x_i)\right| \leq \epsilon.$$

Since $|m(V_i)f(x_i)| \leq \|f\|_{B,p} \cdot \|m\|_p$, it follows that

$$\left|\int f\,dm\right| \leq \max\{\|f\|_{B,p} \cdot \|m\|_p, \epsilon\},$$

for each $\epsilon > 0$, and the proof is complete.

We denote by τ_b the topology on $C(X,E)$ of uniform convergence on the bounding subsets of X.

Lemma 6.2 *The space $S(X,E)$ is τ_b-dense in $C(X,E)$.*

Proof: Let $f \in C(X,E)$, $p \in cs(E)$, $\epsilon > 0$ and B a bounding subset of X. There are $x_1, \dots, x_n$ in B such that the sets

$$V_k = \{x : p(f(x) - f(x_k)) \leq \epsilon\}, \quad k = 1, \dots, n,$$

are pairwise disjoint and cover B. If $g = \sum_{k=1}^{n} \chi_{V_k} f(x_k)$, then $\|f - g\|_{B,p} \leq \epsilon$ and the Lemma follows.

Theorem 6.3 *For $m \in M_b(X, E')$, let*

$$\psi_m : C(X,E) \to \mathbb{K}, \quad \psi_m(f) = \int f \, dm.$$

Then ψ_m is τ_b-continuous and $M_b(X,E')$ is algebraically isomorphic to the dual space of $(C(X,E), \tau_b)$ via the isomorphism $m \mapsto \psi_m$.

Proof: In view of Theorem 6.1, ψ_m is an element of $G = (C(X,E), \tau_b)'$. On the other hand, let $\psi \in G$. Since $\tau_b|_{C_{rc}(X,E)}$ is coarser than the topology τ_u of uniform convergence, there exists $m \in M(X,E')$ such that $\psi(f) = \int f \, dm$ for all $f \in C_{rc}(X,E)$. Let B a bounding subset of X and $p \in cs(E)$ be such that

$$\{f \in C(X,E) : \|f\|_{B,p} \leq 1\} \subset \{f : |\psi(f)| \leq 1\}.$$

It follows that B is a support set for m and so $m \in M_b(X,E')$. Now ψ and ψ_m are both τ_b-continuous and they coincide on the τ_b-dense subspace $S(X,E)$ of $C(X,E)$. Thus $\psi = \psi_m$ and the result follows.

Theorem 6.4 *Let $m \in M_b(X,E')$. If $p \in cs(E)$ is such that $\|m\|_p < \infty$, then $m \in \mathcal{M}_{u,p}(X,E')$.*

Proof: Let B be a bounding support for m and let $(V_i)_{i \in I}$ be a clopen partition of X. The set $\overline{B}^{\theta_o X}$ is compact and

$$\overline{B}^{\theta_o X} \subset \theta_o X \subset \bigcup_i \overline{V_i}^{\beta_o X}.$$

Hence, there exists a finite subset J of I such that

$$\overline{B}^{\theta_o X} \subset \bigcup_{i \in J} \overline{V_i}^{\beta_o X}$$

and so $B \subset \bigcup_{i \in J} V_i$, which implies that $m_p(\bigcup_{i \notin J} V_i) = 0$. Thus $m \in \mathcal{M}_{u,p}(X,E')$ by [13], Theorem 5.7.

Theorem 6.5 *The topology induced by τ_b on $C_b(X,E)$ is coarser than β'_u.*

Proof: Let B be a bounding subset of X, $p \in cs(E)$ and $H \in \Omega_u$. There exists a clopen partition $(V_i)_{i \in I}$) of X such that

$$H \subset \beta_o X \setminus \bigcup_{i \in I} \overline{V_i}^{\beta_o X}.$$

As in the proof of the preceding Theorem, there exists a finite subset J of I such that $B \subset \bigcup_{i \in J} V_i = V$. If $h = \chi_V$, then $h^{\beta_o} = \chi_{\overline{V}^{\beta_o X}}$ vanishes on H and

$$\{f \in C_b(X,E) : \|hf\|_p \leq \epsilon\} \subset \{f : \|f\|_{B,p} \leq \epsilon\}$$

which clearly completes the proof.

7 $M_s(X)$ as a Completion

The space $M_s(X)$ was introduced in [12]. It is the space of the so called separable members of $M_\sigma(X)$. For $m \in M(X)$, d a continuous ultrapseudometric on X and A a d-clopen subset of X, we define

$$|m|_d(A) = \sup\{|m(B)| : B \subset A,\ B \quad \mathrm{d}-clopen\}.$$

For $F \subset X$, we define

$$|m|_d^\star(F) = \inf \sup_n |m|_d(A_n),$$

where the infimum is taken over the family of all sequences (A_n) of d-clopen sets which cover F. An element m of $M_\sigma(X)$ is said to be separable if, for each continuous ultrapseudometric d on X, there exists a d-closed, d-separable subset G of X such that $|m|_d^\star(X \setminus G) = 0$. As it is shown in [12], if $m \in M_s(X)$, then every $f \in C_b(X)$ is m-integrable. Let now $G = (C_b(X), \tau_u)'$, where τ_u is the topology of uniform convergence. For each $x \in X$, let δ_x be the corresponding Dirac measure. Thus $\delta_x \in G$, $\delta_x(f) = f(x)$. Let $L(X)$ be the subspace of G spanned by the set $\{\delta_x : x \in X\}$. Let $\mathcal{E}_u$ be the collection of all equicontinuous τ_u-bounded subsets of $C_b(X)$. Consider the dual pair $< C_b(X), L(X) >$.

Lemma 7.1 *If $B \in \mathcal{E}_u$, then the bipolar B^{oo} of B, with respect to the pair $< C_b(X), L(X) >$, is also in $\mathcal{E}_u$.*

Proof: Let $\sigma = \sigma(C_b(X), L(X))$. By [21], Proposition 4.10, we have that $B^{oo} = \left(\overline{co(B)}^{\sigma}\right)^e$, where $co(B)$ is the absolutely convex hull of B , $\overline{co(B)}^{\sigma}$ the σ-closure of $co(B)$ and, for A an absolutely convex subset of a vector space E over $\mathbb{K}$, A^e is the edged hull of A (see [21]). Thus, if $|\lambda| > 1$, we have

$$B^{oo} \subset \lambda \overline{co(B)}^{\sigma}.$$

So it suffices to show that the set $B_1 = \overline{co(B)}^{\sigma}$ is in $\mathcal{E}_u$. But

$$\sup_{F \in B_1} \|f\| = \sup_{f \in B} \|f\| < \infty.$$

Given $x \in X$, and $\epsilon > 0$, there exists a neighborhood V of x such that $|f(x) - f(y)| \leq \epsilon$ for every $f \in B$ and every $y \in V$. It is easy to see , for $f \in B_1$ and $y \in V$, we have $|f(x) - f(y)| \leq \epsilon$. This proves that $B^{oo} \in \mathcal{E}_u$ and the result follows.

Consider now on $L(X)$ the topology e_u of uniform convergence on the members of $\mathcal{E}_u$. Thus e_u is generated by the family of seminorms p_B, $B \in \mathcal{E}_u$, where $p_B(u) = \sup_{f \in B} |u(f)|$. Let

$$\Delta : X \to L(X), \quad x \mapsto \delta_x.$$

Clearly Δ is one-to-one.

Theorem 7.2 *The map*

$$\Delta : X \to (\Delta(X), e_u|_{\Delta(X)})$$

is a homeomorphism.

Proof: Let (x_γ) be a net in X converging to some $x \in X$ and let $B \in \mathcal{E}_u$ and $\epsilon > 0$. There exists a neighborhood V of x such that

$$p_B(\delta_x - \delta_y) = \sup_{f \in B} |f(x) - f(y)| < \epsilon$$

if $y \in V$. Let γ_o be such that $x_\gamma \in V$ if $\gamma \geq \gamma_o$. Now, for $\gamma \geq \gamma_o$, we have that $p_B(\delta_x - \delta_{x_\gamma}) < \epsilon$, which proves that Δ is continuous. Conversely, suppose that for a net (x_γ) in X, we have that $\delta_{x_\gamma} \xrightarrow{e_u} \delta_x$ and let V be a clopen neighborhood of x. Let $f = \chi_V$, $B = \{f\} \in \mathcal{E}_u$. There exists a γ_o such that $p_B(x - x_\gamma) = |f(x) - f(y)| < 1$ when $\gamma \geq \gamma_o$. But then $x_\gamma \in V$ when $\gamma \geq \gamma_o$, which proves that $x_\gamma \to x$, and the result follows.

In view of the preceding Theorem, we may consider X as a topological subspace of $(L(X), e_u)$.

Theorem 7.3 *e_u is the finest of all polar locally convex topologies γ on $L(X)$ which induce on X its topology and for which X is a bounded subset of $(L(X), \gamma)$.*

Proof: The topology e_u is clearly polar. We show first that X is e_u-bounded. Indeed, let $B \in \mathcal{E}_u$ and choose $\lambda \in \mathbb{K}$ with $|\lambda| > \sup_{f \in B} \|f\|$. Since $|\delta_x(f)| \leq |\lambda|$, for all $f \in B$, we have that $X \subset \lambda B^o$, and so X is e_u-bounded. Suppose now that γ is a polar topology on $L(X)$ which induces on X its topology and for which X is γ-bounded. Let W be a polar γ-neighborhood of zero in $L(X)$ and take $B = \{\phi|_X : \phi \in W^o\}$, where W^o is the polar of W in the dual space of $(L(X), \gamma)$. Every $f \in B$ is continuous on X. Since X is γ-bounded, there exists $\lambda \in \mathbb{K}$, such that $X \subset \lambda W$ and so $\sup_{f \in B} \|f\| \leq |\lambda|$. Also, B is an equicontinuous set. In fact, let $x \in X \subset \lambda W$. Let α be a non-zero element of $\mathbb{K}$ and take $V = (x + \alpha W) \cap X$. Then V is a neighborhood of x in X. If $y \in V$, then for $\phi \in W^o$ and $f = \phi|_X$, we have $|fy) - f(x)| \leq |\alpha|$. This proves that $B \in \mathcal{E}_u$. Moreover $B^o \subset W^{oo} = W$, which proves that W is a neighborhood of zero in $L(X)$ for the topology e_u. This completes the proof.

Theorem 7.4 *The dual space of $F = (L(X), e_u)$ coincides with $C_b(X)$.*

Proof: Since e_u is finer than the weak topology $\sigma(L(X), C_b(X))$, it follows that $C_b(X)$ is contained in F' (considering every element of $C_b(X)$ as a linear functional on $L(X)$). On the other hand, let $\phi \in F'$ and define $f : X \to \mathbb{K}$, $\quad f(x) = \phi(\delta_x)$. Then f is continuous. Since X is e_u-bounded, there exists $\lambda \in \mathbb{K}$ such that $X \subset \lambda D$, where $D = \{u \in L(X) : |\phi(u)| \leq 1\}$. It follows that $\|f\| \leq |\lambda|$ and so $f \in C_b(X)$. It is now clear that $\phi(u) =< f, u >$, for all $u \in L(X)$, and the result follows.

Next we will look at the completion $\hat{F}$ of the space $F = (L(X), e_u)$. Since F is a Hausdorff polar space, $\hat{F}$ is the space of all linear functionals on $F' = C_b(X)$ which are $\sigma(C_b(X), L(X))$-continuous on each e_u-equicontinuous subset of $C_b(X)$ (by [16]). We will prove that $\hat{F}$ coincides with the space $M_s(X)$ equipped with the topology of uniform convergence on the members of $\mathcal{E}_u$.

Lemma 7.5 *A subset B of $C_b(X)$ is e_u-equicontinuous iff $B \in \mathcal{E}_u$.*

Proof: If $B \in \mathcal{E}_u$, then B^o is an e_u-neighborhood of zero and so B^{oo} (and hence also its subset B) is e_u-equicontinuous . Conversely, let B be an e_u-equicontinuous subset of $C_b(X)$. There exists $B_1 \in \mathcal{E}_u$ such that $B \subset B_1^{oo}$. Since $B_1^{oo} \in \mathcal{E}_u$, the same holds for B and the Lemma follows.

Theorem 7.6 *The completion of the space $F = (L(X), e_u)$ is the space $M_s(X)$ equipped with the topology of uniform convergence on the members of $\mathcal{E}_u$.*

Proof: Let $u \in \hat{F}$. Then u is a linear functional on $F' = C_b(X)$.
Claim I. u is τ_u-continuous. In fact, Let (f_n) be a sequence in $C_b(X)$ with $f_n \xrightarrow{\tau_u} 0$. The set $B = \{f_n : n \in \mathbf{N}\}$ belongs to $\mathcal{E}_u$ and $f_n \to 0$ in the weak topology $\sigma(C_b(X), L(X))$. Since $u \in \hat{F}$, we have that $u(f_n) \to 0$, which proves that u is τ_u-continuous.
Claim II. u is β_u-continuous. To prove this, it suffices to show that, on every member of $\mathcal{E}_u$, u is continuous with respect to the topology of simple convergence (by [12], Theorem 6.4). But the last topology coincides with $\sigma(C_b(X), L(X))$. Hence the claim follows.
By [12], Theorem 6.4, there exists an $m \in M_s(X)$ such that $u(f) = \int f\, dm$, for all $f \in C_b(X)$. Conversely, if $m \in M_s(X)$, then the linear functional u_m on $C_b(X)$, $u_m(f) = \int f\, dm$, is in $\hat{F}$ by Lemma 7.5 and by [12], Theorem 6.4.This clearly completes the proof.

Theorem 7.7 *Let E be a Hausdorff polar locally convex space and let $f : X \to E$ be continuous such that $f(X)$ is bounded. Then there exists a unique continuous linear map $T : (L(X), e_u) \to E$ such that $T = f$ on X. If E is in addition complete, then there exists a continuous linear map $T : (M_s(X), e_u) \to E$ such that $T = f$ on X.*

Proof: Let $T : (L(X), e_u) \to E$ be the unique continuous linear extension of f. We need to show that T is e_u-continuous. Let τ_o be the polar topology of E. Then $\tau_1 = T^{-1}(\tau_o)$ is polar and so the supremum $\tau_2 = e_u \vee \tau_1$ is polar. It is easy to see that X is τ_2-bounded. Also $\tau_2|_X$ coincides with the topology of X. In view of Theorem 7.3, τ_2 coincide with e_u which clearly implies that T is e_u-continuous. In case E is complete, T has a continuous linear extension $\hat{T} : (M_s(X), e_u) \to E$ since $(L(X), e_u)$ is a dense topological subspace of $(M_s(X), e_u)$. Hence the result follows.

A linear functional ϕ on $C_b(X)$ is said to be bounded if it is τ_u-continuous. Equivalently, ϕ is bounded if

$$\|\phi\| = \sup\{|\phi(f|/\|f\| : f \in C_b(X), f \neq 0\} < \infty.$$

Theorem 7.8 *For a linear functional ϕ on $C_b(X)$ the following are equivalent :*

1. *There exists $m \in M_s(X)$ such that $\phi(f) = \int f\,dm$ for all $f \in C_b(X)$.*

2. *ϕ is bounded and, for each equicontinuous net (f_δ) in $C_b(X)$, with $f_\delta \downarrow 0$, we have that $\phi(f_\delta) \to 0$.*

Proof: (1) $\Rightarrow$ (2) . Let $m \in M_s(X)$ be such that $\phi = u_m$, $u_m(f) = \int f\,dm$. By Theorem 7.6, ϕ belongs to the completion of $F = (L(X), e_u)$. Then ϕ is bounded. Let $(f_\delta)_{\delta\in\Delta}$ be an equicontinuous net with $f_\delta \downarrow 0$. If $\delta_o \in \Delta$, then taking the subnet $(f_\delta)_{\delta\geq\delta_o}$ we see that $\{f_\delta : \delta \geq \delta_o\} \in \mathcal{E}_u$. Since $f_\delta(x) \to 0$ for all x, we have that $\phi(f_\delta) \to 0$.
(2) $\Rightarrow$ (1). Since ϕ is bounded, there exists an $m \in M(X)$ such that $\phi(f) = \int f\,dm$ for all $f \in C_{rc}(X)$.
Claim I. $m \in M_s(X)$. Indeed, let $(V_i)_{i\in I}$ be a clopen partition of X. For each finite subset J of I, let $A_J = \bigcup_{i\in J} V_i$, $B_J = A_J^c$. If $f_J = \chi_{B_J}$, then $f_J \downarrow 0$. Also (f_J) is equicontinuous and $f_J \to 0$ pointwise. By our hypothesis, $m(B_J) = \phi(f_J) \to 0$. Thus $m(X) - \sum_{i\in J} m(V_i) = m(B_J) \to 0$, and so $m \in M_s(X)$ by [12], Theorem 6.9.
Claim II. $\phi = u_m$. Indeed, let $f \in C_b(X)$ and $\epsilon > 0$. consider the equivalence relation $\sim$ on X, $x \sim y$ iff $|f(x) - f(y)| \leq \epsilon$. Let $(V_i)_{i\in I}$ be the clopen partition of X corresponding to $\sim$. Let $x_i \in V_i$, $\alpha_i = f(x_i)$. For each finite subset J of I, let $g_J = \sum_{i\in J} \alpha_i \chi_{V_i}$, $h_J = \sum_{i\notin J} \alpha_i \chi_{V_i}$. Then (h_J) is equicontinuous and $h_J \downarrow 0$. By our hypothesis, $\phi(h_j) \to 0$. Also, $u_m(h_J) \to 0$. Hence there exists J such that $|u_m(h_J)| < \epsilon$, $|\phi(h_j)| < \epsilon$. Let $g = f - g_J - h_J$. Then $\|g\| \leq \epsilon$. Hence

$$|\phi(g)| \leq \|\phi\| \cdot \|g\| \leq \epsilon\|\phi\|, \quad |u_m(g)| \leq \epsilon\|m\|.$$

Since $\phi(g_J) = u_m(g_J)$, it follows that

$$|\phi(f) - u_m(f)| \leq \max\{\epsilon\|\phi\|, \quad \epsilon\|m\|\}.$$

As $\epsilon > 0$ was arbitrary, we conclude that $\phi(f) = u_m(f)$ and the proof is complete.

For d a bounded continuous ultrapseudometric on X, let

$$\pi_d : X \to X_d, \quad x \mapsto \tilde{x}_d,$$

be the quotient map and let

$$T_d : (C_b(X_d), \beta) \to (C_b(X), \beta_e)$$

be the induced linear map. The dual of the space $(C_b(X), \beta_e)$ is the space $M_s(X)$ (see [12], Theorem 6.4) and

$$T_d^*(M_s(X)) \subset M_\tau(X_d) = M_s(X_d).$$

Lemma 7.9 *The map*

$$T_d^* : (M_s(X), e_u) \to (M_\tau(X_d), e_u)$$

is continuous.

Proof: It follows from the fact that, if $A \in \mathcal{E}_u(X_d)$, then $B = T_d(A) \in \mathcal{E}_u(X)$ and $T_d^\star(B^o) \subset A^o$.

Theorem 7.10 *$(M_s(X), e_u)$ is the projective limit of the spaces $(M_\tau(X_d), e_u)$, with respect to the maps $T_d^\star$, where d ranges over the family of all bounded continuous ultrapseudometrics on X.*

Proof: We need to show that the topology e_u is the weakest of all locally convex topologies τ on $M_s(X)$ for which each

$$T_d^\star : (M_s(X), \tau) \to (M_\tau(X_d), e_u)$$

is continuous. Let τ be such a topology and let $B \in \mathcal{E}_u(X)$. Define $d(x,y) = \sup_{f\in B} |f(x) - f(y)|$. Then d is a bounded continuous ultrapseudometric on X. For each $f \in B$, the function

$$\tilde{f} : X_d \to \mathbb{K}, \quad \tilde{f}(\tilde{x}_d) = f(x),$$

is well defined and continuous. Clearly the set $A = \{\tilde{f} : f \in B\}$ is uniformly bounded. Let $\tilde{x}_d \in X_d$ and $\epsilon > 0$. The set

$$V = \{\tilde{y}_d : \tilde{d}(\tilde{x}_d, \tilde{y}_d) \le \epsilon\}$$

is a neighborhood of $\tilde{x}_d$ and, for $\tilde{y}_d \in V$ and $f \in B$, we have

$$|\tilde{f}(\tilde{y}_d) - \tilde{f}(\tilde{x}_d)| \le \tilde{d}(\tilde{x}_d, \tilde{y}_d) \le \epsilon.$$

Thus $A \in \mathcal{E}_u(X_d)$. Since $T_d^\star$ is τ-continuous, the set $M = (T_d^\star)^{-1}(A^o)$ is a τ-neighborhood of zero. But $M \subset B^o$. Thus B^o is a τ-neighborhood of zero, which proves that τ is finer than e_u. Hence the result follows.

8 $M_{sv_o}(X)$ as a Completion

For each $x \in X$, δ_x may be considered as an element of the algebraic dual $C(X)^\star$ of the space $C(X)$. Let $L(X)$ be the subspace of $C(X)^\star$ spanned by the set $\{\delta_x : x \in X\}$. Let $\mathcal{E} = \mathcal{E}(X)$ be the family of all pointwise bounded equicontinuous subsets of $C(X)$.

Lemma 8.1 *The bidual B^{oo}, of a set $B \in \mathcal{E}$, with respect to the pair $< C(X), L(X) >$, is also in $\mathcal{E}$.*

Proof: The proof is analogous to the one of Lemma 7.1.

Consider on $L(X)$ the locally convex topology E of uniform convergence on the members of $\mathcal{E}$. As in Theorem 7.2, we have the following

Theorem 8.2 *If $\Delta : X \to L(X)$, $\quad x \mapsto \delta_x$, then the map*

$$\Delta : X \to (\Delta(X),\ e|_{\Delta(X)})$$

is a homeomorphism.

In view of the preceding Theorem, we may consider X as a topological subspace of $(L(X), e)$.

Theorem 8.3 *e is the finest of all polar topologies on $L(X)$ which induce on X its topology.*

Proof: The proof is analogous to the one of Theorem 7.3.

The proof of the following Theorem is analogous to the one of Theorem 7.4.

Theorem 8.4 *The dual space of $G = (L(X), e)$ coincides with $C(X)$.*

Lemma 8.5 *A subset B, of the dual space $C(X)$ of $G = (L(X), e)$, is e-equicontinuous iff $B \in \mathcal{E}$.*

Proof: The proof is analogous to that of Lemma 7.5.

Next we will look at the completion of the space $G = (L(X), e)$. Since G is Hausdorff and polar, its completion $\hat{G}$ coincides with the space of all linear functionals on $G' = C(X)$ which are $\sigma(C(X), L(X))$-continuous (equivalently continuous with respect to the topology of simple convergence on e-equicontinuous subsets of $C(X)$, i.e. on the members of $\mathcal{E}$. The topology of $\hat{G}$ is that of uniform convergence on the members of $\mathcal{E}$. Let $M_{sv_o}(X)$ be the space of all $m \in M_s(X)$ for which $supp(m^{\beta_o}) \subset v_oX$. For $m \in M_{sv_o}(X)$, we will show that every $f \in C(X)$ is m-integrable . Thus m defines a linear functional u_m on $C(X)$, $u_m(f) = \int f\, dm$. We will prove that $M_{sv_o}(X)$ is algebraically isomorphic to $\hat{G}$ via the isomorphism $m \mapsto u_m$.

Theorem 8.6 *If $m \in M_b(X)$, then $u_m \in \hat{G}$.*

Proof: Let D be a bounding subset of X which is a support set for m. The set $Z = \bar{D}^{\beta_o X}$ is contained in θ_oX. Let $B \in \mathcal{E}$ and let (f_δ) be a net in B which converges pointwise to the zero function. Since the set $B^{\theta_o} = \{f^{\theta_o} : f \in B\}$ is in $\mathcal{E}(\theta_oX)$ (by Theorem 5.10), given $z \in Z$ and $\epsilon > 0$, there exists a clopen neighborhood W_z of z in θ_oX such that $|f^{\theta_o}(z) - f^{\theta_o}(y)| \le \epsilon/\|m\|$ for all $f \in B$ and all $y \in W_z$. In view of the compactness of Z, there are $z_1, \dots, z_n$ in Z such that $Z \subset \bigcup_{k=1}^n W_{z_k}$. Let $V_k = X \cap W_{z_k}$. If $a, b \in V_k$, then $|f(a) - f(b)| \le \epsilon/\|m\|$ for all $f \in B$. Let $A_1 = V_1$, $\quad A_{k+1} = V_{k+1} \setminus \bigcup_{i=1}^k V_k$, for $k = 1, \dots, n-1$. Keeping those A_i which are not empty, we may assume that $A_i \neq \emptyset$ for all i. Choose $x_i \in A_i$. Clearly $|m|(X \setminus \bigcup_{k=1}^n A_k) = 0$. Since $f_\delta \to 0$ pointwise, there exists δ_o such that

$$\max\{|f_\delta(x_k)| : 1 \le k \le n\} \le \epsilon/\|m\|$$

for all $\delta \ge \delta_o$. Let now $\delta \ge \delta_o$. Then

$$\left|\int_{A_k} f_\delta\, dm - m(A_k) f_\delta(x_k\right| \le \epsilon \quad \textit{and} \quad |m(A_k) f_\delta(x_k)| \le \epsilon,$$

which implies that $|\int_{A_k} f_\delta \, dm| \le \epsilon$. Thus, for $\delta \ge \delta_o$, we have

$$\left|\int f_\delta \, dm\right| = \left|\sum_{k=1}^{n} \int_{A_k} f_\delta \, dm\right| \le \epsilon,$$

which completes the proof.

Theorem 8.7 *Let $m \in M_{sv_o}(X)$, $g \in C(X)$ and d a continuous ultrapseudometric on X be such that g is d-uniformly continuous. Then :*

1. *g is m-integrable.*
2. *If $\mu = T_d^\star m \in M_\tau(X_d)$, then μ has compact support.*
3. *The function*
$$\tilde{g} : X_d \to \mathbb{K}, \quad \tilde{g}(\tilde{x}_d) = g(x),$$
is well defined and continuous. Moreover $\int \tilde{g} \, d\mu = \int g \, dm$.
4. *$u_m \in \hat{G}$.*

Proof: (1). Let $V_n = \{x \in X : |g(x)| \le n\}$, $W_n = V_n^c$. Since $W_n \downarrow 0$ and $supp(m^{\beta_o}) \subset \upsilon_o X$, there exists n such that $|m|(W_n) = 0$ (by Theorem 2.4). Let $h = g \cdot \chi_{V_n}$. Then $f = h$ m.a.e. (see [14, Definition 2.4]), and so f is m-integrable since h is m-integrable. Moreover $\int g \, dm = \int h \, dm$.
(2) Since μ is τ-additive, we have

$$supp(\mu^{\beta_o}) = \overline{supp(\mu)}^{\beta_o X_d}.$$

Now it suffices to show that $supp(\mu)$ is bounding since X_d is a μ_o-space. So we need to prove that $supp(\mu^{\beta_o}) \subset \upsilon_o X_d$. To show this it is enough to prove that

$$supp(\mu^{\beta_o}) \subset \pi^{\beta_o}(supp(m^{\beta_o})) = D,$$

where $\pi : X \to X_d$ is the quotient map. So, let W be a clopen subset of $\beta_o X$ which is disjoint from D. Then $(\pi^{\beta_o})^{-1}(W)$ is disjoint from $supp(m^{\beta_o})$ and

$$\begin{aligned}\mu^{\beta_o}(W) &= \mu(W \cap X_d) = < T_d^\star m, \chi_{W \cap X_d} > \\ &= m(\pi^{-1}(W \cap X_d)) = m^{\beta_o}\left(\overline{\pi^{-1}(W \cap X_d)}^{\beta_o X}\right).\end{aligned}$$

But

$$\pi^{-1}(W \cap X_d) \subset (\pi^{\beta_o})^{-1}(W) \quad \text{and so} \quad \overline{\pi^{-1}(W \cap X_d)}^{\beta_o X} \subset (\pi^{\beta_o})^{-1}(W)$$

which implies that $\mu^{\beta_o}(W) = 0$. It follows that the support of μ^{β_o} is contained in D and this proves (2).
(3). It is easy to see that $\tilde{g}$ is well defined and continuous. Let

$$A_n = \{x \in X : |g(x)| \le n\}.$$

There exists an n such that $|m|(A_n^c) = 0$. If $h = g \cdot \chi_{A_n}$, then $\pi(A_n)$ is d-clopen and $\tilde{h} = \tilde{g} \cdot \chi_{\pi(A_n)}$. If Y is a clopen subset of X_d disjoint from $\pi(A_n)$, then $\mu(Y) = m(\pi^{-1}(Y)) = 0$ since $\pi^{-1}(Y)$ is disjoint from A_n. Thus

$$\int g\, dm = \int h\, dm = \int \tilde{h}\, d\mu = \int \tilde{g}\, d\mu.$$

(4). Let $B \in \mathcal{E}$ and let (f_δ) be a net in B which converges pointwise to the zero function. Define $d(x,y) = \sup_{f\in B} |f(x) - f(y)|$. Now $\tilde{B} = \{\tilde{f} : f \in B\} \in \mathcal{E}(X_d)$ and $\tilde{f}_\delta \to 0$ pointwise. Since μ has a bounding support, we have that $\int f_\delta\, dm = \int \tilde{f}_\delta\, d\mu \to 0$ by the preceding Theorem. This proves that $u_m \in \tilde{G}$ and the result follows.

Theorem 8.8 *If $\phi \in \hat{G}$, then there exists an $m \in M_{sv_o}(X)$ such that $\phi = u_m$.*

Proof: Lt $B \in \mathcal{E}_u$ and let (f_δ) be a net in B which converges pointwise to the zero function. Then $\phi(f_\delta) \to 0$, which proves that $\phi|_{C_b(X)}$ belongs to the completion of the space $F = (L(X), e_u)$. Thus, by Theorem 7.6, there exists $m \in M_s(X)$ such that $\phi(f) = \int f\, dm$ for all $f \in C_b(X)$. We will show first that $supp(m^{\beta_o}) \subset v_oX$. In fact, assume that there exists a $z \in supp(m^{\beta_o}) \setminus v_oX$. Let (V_n) be a sequence of clopen subsets of X, with $V_n \downarrow \emptyset$ and $z \in \overline{V_n}^{\beta_o X}$ for all n. Since $z \in supp(m^{\beta_o})$, there exists a clopen subset A_n of $\overline{V_n}^{\beta_o X}$ with $m^{\beta_o}(A_n) = \alpha_n \neq 0$. Let $B_n = A_n \cap X$ and $f_n = \alpha_n^{-1}\chi_{B_n}$. Given $x \in X$, there exists n_o such that $x \notin V_{n_o}$. For $y \notin V_{n_o}$, we have $f_n(y) = 0$ for all $n \geq n_o$. Hence $(f_n) \in \mathcal{E}$ and $f_n \to 0$ pointwise. Thus

$$1 = \alpha_n^{-1} m(B_n) = \int f_n\, dm \to 0,$$

a contradiction. This proves that $m \in M_{sv_o}(X)$. We will finish the proof by showing that $\phi(f) = \int f\, dm$ for all $f \in C(X)$. So, let $f \in C(X)$. For each positive integer n, let

$$A_n = \{x : |f(x)| \geq n\}, \quad f_n = f \cdot \chi_{A_n}, \quad g_n = f - f_n.$$

Then $(f_n) \in \mathcal{E}$ and $f_n \to 0$ pointwise. Thus $\phi(f_n) \to 0$ and $u_m(f_n) \to 0$. Also, $\phi(g_n) = u_m(g_n)$. It follows that $\phi(f) - u_m(f) = 0$, which completes the proof.

Combining Theorems 8.7 and 8.8, we get

Theorem 8.9 *The completion of the space $G = (L(X), e)$ coincides with the space $M_{sv_o}(X)$ equipped with the topology of uniform convergence on the members of $\mathcal{E}$.*

By Theorem 8.6, $M_b(X)$ is a subspace of $M_{sv_o}(X)$. We will denote also by e the topology on $M_b(X)$ of uniform convergence on the members of $\mathcal{E}$. For d a continuous ultrapseudometric on X, let $\pi_d : X \to X_d$ be the quotient map and let $S_d : C(X_d) \to C(X)$ be the induced linear map. As it is shown in Theorem 8.7, if $m \in M_{sv_o}(X)$, them $S_d^* m \in M_c(X_d)$.

Lemma 8.10 *For each continuous ultrapseudometric d on X, the map*

$$S_d^* : (M_{sv_o}(X), e) \to (M_c(X_d), e)$$

is continuous.

Proof: Let $A \in \mathcal{E}(X_d)$, $B = S_d(A)$. Then $B \in \mathcal{E}(X)$. If B^o is the polar of B in $M_{sv_o}(X)$ and A^o the polar of A in $M_b(X_d) = M_c(X_d)$, then $S_d^\star(B^o) \subset A^o$ and the result follows.

Theorem 8.11 *$(M_{sv_o}(X), e)$ is the projective limit of the spaces $(M_c(X_d), e)$, with respect to the maps $S_d^\star$, where d ranges over the family of all continuous ultrapseudometrics on X.*

Proof: We need to show that e is the weakest of all locally convex topologies τ on $M_{sv_o}(X)$ for which each of the maps

$$S_d^\star : (M_{sv_o}(X), \tau) \to (M_c(X_d), e)$$

is continuous. So, let τ be such a topology and let $B \in \mathcal{E}(X)$. Define

$$d(x, y) = \sup_{f \in B} |f(x) - f(y)|.$$

Then d is a continuous ultrapseudometric on X. For each $f \in B$, the function

$$\tilde{f} : X_d \to \mathbb{K}, \quad \tilde{f}(\tilde{x}_d) = f(x)$$

is well defined and continuous. Clearly the set $A = \{\tilde{f} : f \in B\}$ is in $\mathcal{E}(X_d)$. Since $S_d^\star$ is τ-continuous, the set $M = (S_d^\star)^{-1}(A^o)$ is a τ-neighborhood of zero. But $M \subset B^o$. Thus B^o is a τ-neighborhood of zero, which proves that τ is finer that e. Hence the result follows.

Theorem 8.12 *For an $m \in M(X)$, the following are equivalent:*

1. *$m \in M_{sv_o}(X)$.*
2. *For each continuous ultrapseudometric d on X the measure*

$$m_d : K(X_d) \to \mathbb{K}, \quad m_d(A) = m(\pi_d^{-1}(A))$$

has compact support.

3. *For each clopen partition $(A_i)_{i \in I}$ of X, there exists a finite subset J_o of I such that $m(\bigcup_{i \notin J} A_i) = 0$ for all finite subsets J of I which contain J_o.*

Proof: (1) $\Rightarrow$ (2). It follows from the fact that $m_d = S_d^\star m$.
(2) $\Rightarrow$ (3). Let $(A_i)_{i \in I}$ be a clopen partition of X and take $f_i = \chi_{A_i}$. If $B_i = \pi_d(A_i)$, then $(B_i)_{i \in I}$ is a clopen partition of X_d. Let Z be a compact support of m_d. There exists a finite subset J_o of I such that $Z \subset \bigcup_{i \in J_o} B_i$. Let the finite subset J of I contain J_o. If $A = \bigcup_{i \notin J} A_i$ and $B = \pi_d(A)$, then $0 = m_d(B) = m(\pi_d^{-1}(B)) = m(A)$.
(3) $\Rightarrow$ (1). Let $(A_i)_{i \in I}$ be a clopen partition of X and let J_o be as in (3). Clearly $m(A_i) = 0$ for all $i \notin J_o$. Thus

$$m(X) = m\left(\bigcup_{i \in J_o} A_i\right) + m\left(\bigcup_{i \notin J_o} A_i\right) = \sum_{i \in J_o} m(A_i) = \sum_{i \in I} m(A_i),$$

and so $m \in M_s(X)$ by [12], Theorem 6.9. To show that

$$supp(m^{\beta_o})) \subset \upsilon_o X$$

it suffices, by Theorem 2.4, to show that if (W_n) is a sequence of clopen subsets of X, with $W_n \downarrow \emptyset$, then there exists n_o such that $m(W_n) = 0$ if $n \geq n_o$. Given such a sequence, let $D_1 = W_1^c$, $D_{n+1} = W_n \setminus W_{n+1}$ for $n \geq 1$. Then (D_n) is a clopen partition of X. By our hypothesis, there exists n_o such that $m(\bigcup_{n \geq n_1} D_n) = 0$ if $n_1 \geq n_o$. For each n, we have $W_n = \bigcup_{k>n} D_k$. Hence, for $n \geq n_o$, we have $m(W_n) = 0$, which completes the proof.

9 Polarly Barrelled Spaces of Continuous Functions

Definition 9.1 *A Hausdorff locally convex space E is called :*

1. *polarly barrelled if every bounded subset of $E'_\sigma = (E', \sigma(E', E))$ is equiconinuous.*
2. *polarly quaasi-barrelled if every strongly bounded subset of E' is equicontinuous.*

We will denote by $C_c(X, E)$ the space $C(X, E)$ equipped with the topology of uniform convergence on compact subsets of X. By $M_c(X, E')$ we will denote the space of all $m \in M(X, E')$ with compact support. The dual space of $C_c(X, E)$ coincides with $M_c(X, E')$.
Recall that a zero-dimensional Hausdorff topological space X is called a μ_o-space (see [1]) if every bounding subset of X is relatively compact. We denote by $\mu_o X$ the smallest of all μ_o-subspaces of $\beta_o X$ which contain X. Then $X \subset \mu_o X \subset \theta_o X$ and, for each bounding subset A of X, the set $\overline{A}^{\beta_o X}$ is contained in $\mu_o X$ (see [1]). Moreover, if Y is another Hausdorff zero-dimensional space and $f : X \to Y$, then $f^{\beta_o}(X) \subset \mu_o Y$ and so there exists a continuous extension $f^{\mu_o} : \mu_o X \to \mu_o Y$ of f.

Theorem 9.2 *Assume that $E' \neq \{0\}$ and let $G = C_c(X, E)$. Then G is polarly barrelled iff X is a μ_o-space and E polarly barrelled.*

Proof: Assume that G is polarly barrelled.
I. E is polarly barrelled. Indeed, let Φ be a $w^\star$-bounded subset of E' and let $x \in X$. For $u \in E'$, let

$$u_x : G \to \mathbb{K}, \quad u_x(f) = u(f(x)).$$

Let $H = \{u_x : u \in \Phi\}$. For $f \in C(X, E)$, we have

$$\sup_{u \in \Phi} |u_x(f)| = \sup_{u \in \Phi} |u(f(x))| < \infty$$

and so H is a $w^\star$-bounded subset of G'. By our hypothesis, there exists $p \in cs(E)$ and Y a compact subset of X such that

$$\{f \in G : \|f\|_{Y,p} \leq 1\} \subset H^o.$$

But then $\{s \in E : p(s) \leq 1\} \subset \Phi^o$ and so Φ is equicontinuous.
II. X is a μ_o-space. In fact, let A be a bounding subset of X and let $x' \in E'$, $x' \neq 0$. Define p on E by $p(x) = |x'(s)|$. Then $p \in cs(E)$. The set

$$D = \{f \in G : \|f\|_{A,p} \leq 1\}$$

is a polar barrel in G and so D is a neighborhood of zero in G. Let Y a compact subset of X and $q \in cs(E)$ be such that

$$\{f \in G : \|f\|_{Y,p} \leq 1\} \subset D.$$

But then $A \subset Y$ and so $\overline{A}$ is compact.
Conversely, suppose that E is polarly barrelled and X a μ_o-space. Let H be a $w^\star$-bounded subset of the dual space $M_c(X, E')$ of G. Let $s \in E$ and

$$D = \{ms : m \in H\} \subset M(X).$$

For $h \in C_{rc}(X)$, we have that

$$\sup_{m \in H} | < ms, h > | = \sup_{m \in H} | < m, hs > | < \infty.$$

Thus, considering $M(X)$ as the dual of the Banach space $F = (C_{rc}(X), \tau_u)$, D is $w^\star$-bounded of F' and so $\sup_{m \in H} \|ms\| = d_s < \infty$. Hence, $|m(V)s| \leq d_s$ for all $V \in K(X)$. It follows that the set

$$M = \bigcup_{m \in H} m(K(X))$$

is a $w^\star$-bounded subset of E'. Since E is polarly barrelled, there exists $p \in cs(E)$ such that $|u(s)| \leq 1$ for all $u \in M$ and all $s \in E$ with $p(s) \leq 1$. Hence $\sup_{m \in H} \|m\|_p < \infty$. We may choose p so that $\|m\|_p \leq 1$ for all $m \in H$. Let

$$Z = S(H) = \overline{\bigcup_{m \in H} supp(m)}.$$

Then Z is bounding. In fact, assume that Z is not bounding. Then, by [11], Proposition 6.6, there exists a sequence (m_n) in H and $f \in C(X, E)$ such that $< m_n, f >= \lambda^n$, for all n, where $|\lambda| > 1$, which contradicts the fact that H is $w^\star$-bounded. By our hypothesis now, Z is compact. Since

$$\{f \in G : \|f\|_{Z,p} \leq 1\} \subset H^o,$$

the result follows.

Corollary 9.3 *$C_c(X)$ is polarly barrelled iff X is a μ_o-space.*

Let now G, E be Hausdorff locally convex spaces. We denote by $L_s(G, E)$ the space $L(G, E)$ of all continuous linear maps, from G to E, equipped with the topology of simple convergence.

Theorem 9.4 *Assume that E is polar and let G be polarly barrelled. If E is a μ_o-space (e.g. when E is metrizable or complete), then $L_s(G,E)$ is a μ_o-space.*

Proof: Let Φ be a bounding subset of $L_s(G,E)$. For $x \in G$, the set

$$\Phi(x) = \{\phi(x) : \phi \in \Phi\}$$

is a bounding subset of E and hence its closure M_x in E is compact. Φ is a topological subspace of E^G and it is contained in the compact set $M = \prod_{x \in G} M_x$. Since the closure of Φ in E^G is compact, it suffices to show that this closure is contained in $L(G,E)$. To this end, we prove first that, given a polar neighborhood W of zero in E, there exists a neighborhood U of zero in G such that $\phi(U) \subset W$ for all $\phi \in \Phi$. In fact, for $\phi \in \Phi$, let ϕ' be the adjoint map. Let

$$Z = \bigcup_{\phi \in \Phi} \phi'(H),$$

where H is the polar of W in E'. If $x \in G$, then $\Phi(x)$ is a bounded subset of E and hence $\Phi(x) \subset \alpha W$, for some $\alpha \in \mathbb{K}$. If now $\phi \in \Phi$ and $u \in H$, then

$$| < \phi'(u), x > | = | < u, \phi(x > | \leq |\alpha|,$$

which proves that Z is a $w^\star$-bounded subset of G'. As G is polarly barrelled, the polar $U = Z^o$, of Z in G, is a neighborhood of zero and $\phi(U) \subset H^o = W$, for all $\phi \in \Phi$, which proves our claim. Let now $\phi \in E^G$ be in the closure of Φ. Then ϕ is linear. There exists a net (ϕ_δ) in Φ converging to ϕ in E^G. If $x \in U$, then $\phi(x) = \lim \phi_\delta(x) \in W$, which proves that ϕ is continuous. Hence the result follows.

Corollary 9.5 *If E is polarly barrelled, then the weak dual E'_σ of E is a μ_o-space.*

Theorem 9.6 *Suppose that E is polar and G polarly barrelled. For $f \in C(X,E)$, let $f^{\mu_o} : \mu_o X \to \hat{E}$ be its continuous extension. If $T : G \to C_c(X,E)$ is a continuous linear map, then the map*

$$\tilde{T} : G \to C_c(\mu_o X, \hat{E}), \quad s \mapsto (Ts)^{\mu_o},$$

is continuous

Proof: Note that $\hat{E}$ is θ_o-complete and hence a μ_o-space. Let

$$\phi : X \to L_s(G,E), \quad < \phi(x), s >= (Ts)(x).$$

Then ϕ is continuous. Since $L_s(G,\hat{E})$ is a μ_o-space, there exists a continuous extension

$$\phi^{\mu_o} : \mu_o X \to L_s(G,\hat{E}).$$

Let now A be a compact subset of $\mu_o X$ and p a polar continuous seminorm on E. We denote also by p the continuous extension of p to all of $\hat{E}$. Let

$$V = \{g \in C(\mu_o X, \hat{E}) : \|g\|_{A,p} \leq 1\}.$$

The set $\Phi = \phi^{\mu_o}(A)$ is compact in $L_s(G, \hat{E})$. As in the proof of Theorem 9.4, there exists a neighborhood U of zero in G such that

$$\psi(U) \subset W = \{s \in \hat{E} : p(s) \leq 1\},$$

for all $\psi \in \Phi$. Now, for $y \in A$ and $s \in U$, we have

$$p((\tilde{T}s)(y)) = p(< \phi^{\mu_o}(y), s >) \leq 1$$

and so $\tilde{T}s \in V$. This proves that $\tilde{T}$ is continuous and the result follows.

Theorem 9.7 *Assume that E is polar and polarly barrelled and let τ_o be the locally convex topology on $C(X, E)$ generated by the seminorms $f \mapsto \|f^{\mu_o}\|_{A,p}$, where A ranges over the family of all compact subsets of $\mu_o X$ and $p \in cs(E)$. Then :*

1. *$(C(X, E), \tau_o)$ is polarly barrelled and τ_o is finer than τ_b (and hence finer than τ_c).*
2. *If τ is any polarly barrelled topology on $C(X, E)$ which is finer than τ_c, then τ is finer than τ_o. Hence τ_o is the polarly barrelled topology associated with each of the topologies τ_b and τ_c.*

Proof: (1). Since E is polarly barrelled, the same is true for $\hat{E}$. The space $F = C_c(\mu_o X, \hat{E})$ is polarly barrelled and the map

$$S : (C(X, E), \tau_o) \to F, \quad f \mapsto f^{\mu_o},$$

is a linear homeomorphism. Thus τ_o is polarly barrelled. Also, since for each bounding subset B of X, its closure $\overline{B}^{\mu_o X}$ is compact, it follows that τ_o is finer than τ_b.
(2). Let τ be a polarly barrelled topology on $C(X, E)$, which is finer than τ_c, and let $G = (C(X, E), \tau)$. The identity map

$$T : G \to C_c(X, E)$$

is continuous and hence the map

$$\tilde{T} : G \to C_c(\mu_o X, \hat{E}), \quad f \mapsto f^{\mu_o},$$

is continuous. This proves that τ_o is coarser than τ and the Theorem follows.

Theorem 9.8 *Suppose that E is polar. Then $G = (C(X, E), \tau_b)$ is polarly barrelled iff E is polarly barrelled and, for each compact subset A of $\mu_o X$, there exists a bounding subset B of X such that $A \subset \overline{B}^{\mu_o X}$.*

Proof: Assume that G is polarly barrelled. It is easy to see that E is polarly barrelled. In view of the preceding Theorem, $\tau_b = \tau_o$. Thus, for each compact subset A of $\mu_o X$ and each non-zero $p \in cs(E)$, there exist a bounding subset B of X and $q \in cs(E)$ such that

$$\{f \in C(X, E) : \|f\|_{B,q} \leq 1\} \subset \{f : \|f^{\mu_o}\|_{A,p} \leq 1\}.$$

It follows easily that $A \subset \overline{B}^{\mu_o X}$. Conversely, suppose that the condition is satisfied. The condition clearly implies that τ_o is coarser than τ_b and hence $\tau_b = \tau_o$, which implies that G is polarly barrelled by the preceding Theorem.

Let us say that a family $\mathcal{F}$ of subsets of a a set Z is finite on a subset F of Z if the family of all members of $\mathcal{F}$ which meet F is finite.

Definition 9.9 *A subset D, of a topological space Z, is said to be w-bounded if every family $\mathcal{F}$ of open subsets of Z, which is finite on each compact subset of Z, is also finite on D. If this happens for families of clopen sets, then D is said to be w_o-bounded. We say that Z is a w-space (resp. a w_o-space) if every w-bounded (resp. w_o-bounded) subset is relatively compact.*

Lemma 9.10 *A subset D, of a zero-dimensional topological space Z, is w-bounded iff it is w_o-bounded.*

Proof: Assume that D is not w-bounded. Then, there exists an infinite sequence (x_n) of distinct elements of D and a sequence (V_n) of open sets such that $x_n \in V_n$ and (V_n) is finite on each compact subset of X. By [15, Lemma 2.5], there exists a subsequence (x_{n_k}) and pairwise disjoint clopen sets W_k with $x_{n_k} \in W_k$. We may choose $W_k \subset V_{n_k}$. Now (W_k) is clearly finite on each compact subset of X, which implies that D is not w_o-bounded. Hence the Lemma follows.

We easily get the following

Lemma 9.11 *Every w_o-bounded subset of X is bounding.*

Theorem 9.12 *Assume that $E' \neq \{0\}$. Then $G = C_c(X, E)$ is polarly quasi-barrelled iff E is polarly quasi-barrelled and X a w_o-space.*

Proof: Suppose that E is polarly quasi barrelled and X a w_o-space. Let H be a strongly bounded subset of the dual space $M_c(X, E)$ of G. We show first that there exists $p \in cs(E)$ such that $\sup_{m \in H} \|m\|_p < \infty$. In fact, let B be a bounded subset of E and consider the set

$$D = \{ms : m \in H,\ s \in B\}.$$

If $h \in C_{rc}(X)$, then the set $\{hs : s \in B\}$ is a bounded subset of G and so

$$\sup_{m \in H} \left| \int hs\, dm \right| = \sup_{m \in H} \left| \int h\, d(ms) \right| < \infty.$$

Considering D a a subset of the dual of the Banach space $F = (C_{rc}(X), \tau_u)$, we see that D is a w^*-bounded subset of F' and hence equicontinuous. Thus

$$d = \sup_{m \in H, s \in B} \|ms\| < \infty.$$

Let

$$\Phi = \bigcup_{m \in H} m(K(X)).$$

Then for $A \in K(X)$, $s \in B$, $m \in H$, we have $|m(A)s| \leq \|ms\| \leq d$. Hence Φ is a strongly bounded subset of E'. By our hypothesis, Φ is an equicontinuous subset of E'. Thus, there exists $p \in cs(E)$ such that $|m(A)s| \leq 1$ for all $m \in H$ and all $s \in E$ with $p(s) \leq 1$. It follows from this that $\sup_{m \in H} \|m\|_p = r < \infty$. We may choose p so that $r \leq 1$. Let now

$$Y = S(H) = \overline{\bigcup_{m \in H} supp(m)}.$$

Then Y is w_o-bounded. Assume the contrary. Then, there exists a sequence (V_n) of distinct clopen subsets of X, such that $V_n \cap Y \neq \emptyset$ for all n and (V_n) is finite on each compact subset of X. . For each n there exists $m_n \in H$ with $V_n \cap supp(m_n) \neq \emptyset$. Then $(m_n)_p(V_n) > 0$. There are a clopen subset W_n of V_n and $s_n \in E$, with $p(s_n) \leq 1$, such that $m(W_n)s_n = \gamma_n \neq 0$. Let $|\lambda| > 1$ and take

$$M = \{\gamma_n^{-1} \lambda^n \chi_{W_n} s_n : n \in \mathbf{N}\}.$$

Since (W_n) is finite on each compact subset of X, it follows that M is a bounded subset of G and so M is absorbed by H^o. Let $\lambda_o \neq 0$ be such that $M \subset \lambda_o H^o$. But then

$$1 \geq |\lambda_o^{-1} \gamma_n^{-1} \lambda^n m_n(W_n) s_n| = |\lambda_o^{-1} \lambda^n|$$

for all n, which is a contradiction. So Y is w_o-bounded and hence compact by our hypothesis. Moreover

$$\{f \in G : \|f\|_{Y,p} \leq 1\} \subset H^o.$$

Indeed, let $\|f\|_{Y,p} \leq 1$. The set $V = \{x : p(f(x)) > 1\}$ is disjoint from Y and hence $m_p(V) = 0$ for all $m \in H$. Thus, for $m \in H$, we have

$$\left| \int_V f \, dm \right| \leq \|f\|_p \cdot m_p(V) = 0$$

and so

$$\left| \int f \, dm \right| = \left| \int_{V^c} f \, dm \right| \leq m_p(V^c) \leq 1.$$

Conversely, suppose that G is polarly quasi-barrelled. Let Φ be a strongly bounded subset of E' and let $x \in X$. For $u \in E'$, define u_x on G by $u_x(f) = u(f(x))$. Then $u_x \in G'$. The set $H = \{u_x : u \in \Phi\}$ is a strongly bounded subset of G'. Indeed, let D be a bounded subset of G. Since the set $\{f(x) : f \in D\}$ is a bounded subset of E, we have that

$$\sup_{f \in D, u \in \Phi} |u_x(f)| = \sup_{f \in D, u \in \Phi} |u(f(x))| < \infty.$$

By our hypothesis, H is an equicontinuous subset of G'. Thus, there exists a compact subset Y of X and $p \in cs(E)$ such that

$$\{f \in G : \|f\|_{Y,p} \leq 1\}.$$

But then $\{s \in E : p(s) \leq 1\} \subset \Phi^o$ and so Φ is an equicontinuous subset of E', which proves that E is polarly quasi-barrelled. Finally, let A be a w_o-bounded subset of X and choose a non-zero element x' of E'. Let $p(s) = |x'(s)|$ and consider the set

$$Z = \{f \in G : \|f\|_{A,p} \leq 1\}.$$

Then Z is a polar set. We will show that Z is bornivorous. So, suppose that there exists a bounded subset M of G which is not absorbed by Z. Then, there exists a sequence (f_n) in M, $\|f_n\|_{A,p} > n$. Let

$$V_n = \{x : p(f_n(x)) > n\}.$$

Then V_n intersects A. Since A is w_o-bounded, there exists a compact subset Y of X such that (V_n) is not finite on Y, which is a contradiction since $\sup_{f\in M} \|f\|_{Y,p} < \infty$. This contradiction shows that Z absorbs bounded subsets of G. In view of our hypothesis, there exist a compact subset Y of X and $q \in cs((E)$ such that

$$\{f \in G : \|f\|_{Y,q} \leq 1\},$$

which implies that $A \subset Y$ and so A is relatively compact. This clearly completes the proof.

Corollary 9.13 *1. $C_c(X)$ is polarly quasi-barrelled iff X is a w_o-space.*

2. If $E' \neq \{0\}$, then $C_c(X,E)$ is polarly quasi-barrelled iff both E and $C_c(X)$ are polarly quasi- barrelled.

Definition 9.14 *A subset W, of a locally convex space E, is said to be absolutely bornivorous if it absorbs every subset S of E for which $\sup_{x\in S}|u(x)| < \infty$ for all $u \in W^o$. The space E is said to be polarly absolutely quasi-barrelled if every polar absolutely bornivorous subset of E is a neighborhood of zero.*

Lemma 9.15 *Every absolutely bornivorous subset W, of a locally convex space E, absorbs bounded subsets of E.*

Proof: Let B be a bounded subset of E and suppose that W does not absorb B. Let $|\lambda| > 1$. Since B is not absorbed by W, there exists $u \in W^o$ such that $\sup_{x\in B}|u(x)| = \infty$. Choose a sequence (x_n) in B such that $|u(x_n)| > |\lambda|^n$ for all n. Since B is bounded, we have that $y_n = \lambda^{-n}x_n \to 0$, and so $u(y_n) \to 0$, a contradiction.

Definition 9.16 *A subset A, of a topological space Z, is called aw_o-bounded if it is w_o-bounded in its subspace topology. The space Z is said to be an aw_o-space if every aw_o-bounded set is relatively compact.*

Theorem 9.17 *If D is an absolutely bornivorous subset of $G = C_c(X,E)$ and if $H = D^o$ is the polar of D in the dual space $M_c(X,E')$ of G, then the set*

$$Y = S(H) = \overline{\bigcup_{m\in H} supp(m)}$$

is aw_o-bounded.

Proof: Assume the contrary. Then, there exists a sequence (O_n) of open subsets of X such that $Z_n = O_n \cap Y \neq \emptyset$, $Z_n \neq Z_k$, for $n \neq k$, and (Z_n) is finite on each compact subset of Y. For each n, there exists an $m_n \in H$ with $O_n \cap supp(m_n) \neq \emptyset$. Let W_n be a clopen subset of O_n such that $m_n(W_n) \neq 0$. Choose $s_n \in E$ such that $m_n(W_n)s_n = 1$, and let $|\lambda| > 1$, $h_n = \lambda^n \chi_{W_n} s_n$. Consider the set $F = \{h_n : n \in \mathbb{N}\}$. For each $m \in H$, the sequence (W_n) is finite on the $supp(m)$ and thus $m(W_n) = 0$ finally, which implies that $\sup_n |< m, h_n >| < \infty$ for all $m \in H$. Therefore, there exists $\alpha \neq 0$ such that $F \subset \alpha D$. But then

$$1 \geq |< \alpha^{-1} h_n, m_n >| = |\alpha^{-1}\lambda^n|,$$

for all n, which is impossible. This contradiction completes the proof.

Theorem 9.18 *Assume that $E' \neq \{0\}$. If the space $G = C_c(X, E)$ is polarly absolutely quasi-barrelled, then E is polarly absolutely quasi-barrelled and X an aw_o-space.*

Proof: Let W be a polar absolutely bornivorous subset of E and let W^o be its polar in E'. Let $x \in X$ and, for $u \in E'$, let $u_x \in E'$, $u_x(f) = u(f(x))$. Consider the set $H = \{u_x : u \in W^o\}$, and let $D = H^o$ be its polar in G. Then D is absolutely bornivorous. Indeed, let $M \subset G$ be such that $\sup_{f \in M} |u_x(f)| < \infty$ for all $u \in W^o$. Thus, for $u \in W^o$, we have that $\sup_{f \in M} |u(f(x))| < \infty$. Let $S = \{f(x) : f \in M\}$. Since, for $u \in W^o$, we have that $\sup_{s \in S} |u(s)| < \infty$ and since W is absolutely bornivorous, there exists $\alpha \in \mathbb{K}$ such that $S \subset \alpha W$. But then $M \subset \alpha D$. So, D is an absolutely bornivorous polar subset of G. By our hypothesis, D is a neighborhood of zero in G. Hence, there exist a compact subset Y of X and $p \in cs(E)$ such that

$$\{f \in G : \|f\|_{Y,p} \leq 1\} \subset D,$$

which implies that

$$\{s \in E : p(s) \leq 1\} \subset W^{oo} = W.$$

This proves that E is polarly absolutely quasi-barrelled. To prove that X is an aw_o-space, consider an aw_o-bounded subset A of X, x' a non-zero element of E' and define $p(s) = |x'(s)|$. The set

$$V = \{f \in C(X, E) : \|f\|_{A,p} \leq 1\}$$

is a polar subset of G. Also V is absolutely bornivorous. In fact, let $Z \subset G$ be such that $\sup_{f \in Z} |u(f)| < \infty$ for each $u \in V^o \subset G'$. We claim that V absorbs Z. Assume the contrary and let $|\lambda| > 1$. There exists a sequence (f_n) in Z, $f_n \notin \lambda^n V$. Let

$$V_n = \{x : p(f_n(x)) > |\lambda|^n\}.$$

Then $V_n \cap A \neq \emptyset$. Since A is aw_o-bounded, there exists a compact subset Y of A such that (V_n) is not finite on Y. Let $g_n = f_n|_Y$ and consider the space $F = C(Y, E)$ with the topology of uniform convergence. Let $q \in cs(F)$, $\quad q(g) = \|g\|_p$. Then q is a polar seminorm on F and so the normed space F_q is polar. Since (V_n) is not finite on Y, it follows that $\sup_n q(g_n) = \infty$. Let $\pi : F \to F_q$ be the canonical map and

$\tilde{g}_n = \pi(g_n)$. Then $\sup_n \|\tilde{g}_n\| = \infty$. Since F_q is polar, there exists $\phi \in F_q'$ such that $\sup_n |\phi(\tilde{g}_n| = \infty$. Let $u = \phi \circ \pi$. For $g \in F$, we have

$$|u(g)| = |\phi(\tilde{g})| \leq \|\phi\| \cdot \|g\|_p.$$

Let

$$\omega : C_c(X, E) \to \mathbb{K}, \quad \omega(f) = u(f|_Y).$$

Then $|\omega(f)| \leq \|\phi\| \cdot \|f\|_{Y,p}$ and so $\omega \in G'$. Let $|\gamma| > \|\phi\|$. If $v = \gamma^{-1}\omega$, then $v \in V^o$. But

$$\sup_{f \in Z} |v(f)| \geq |\gamma^{-1}| \cdot \sup_n |u(g_n| = |\gamma^{-1}| \cdot \sup_n |\phi(\tilde{g}_n)| = \infty,$$

a contradiction. This contradiction shows that V absorbs Z and therefore V is an absolutely bornivorous barrel. Thus V is a neighborhood of zero in G. Let K be a compact subset of X and $r \in cs(E)$ be such that

$$\{f \in G : \|f\|_{K,r} \leq 1\} \subset V.$$

Then $A \subset K$ and so A is relatively compact. This clearly completes the proof.

Theorem 9.19 *Assume that $E' \neq \{0\}$. If E is polarly quasi-barrelled, then $G = C_c(X, E)$ is polarly absolutely quasi-barrelled iff X is an aw_o-space.*

Proof: The necessity follows from the preceding Theorem.
Sufficiency : Let D be a polar absolutely bornivorous subset of G and let $H = D^o$ be its polar in G'. By Theorem 9.17, the set

$$Y = S(H) = \overline{\bigcup_{m \in H} supp(m)}$$

ia aw_o-bounded and hence compact. Let

$$\Phi = \bigcup_{m \in H} m(K(X)).$$

Then Φ is a strongly bounded subset of E'. In fact, let B be a bounded subset of E. The set

$$F = \{\chi_A s : A \in K(X), s \in B\}$$

is bounded in G. Since D is bornivorous, there exists a non-zero $\alpha \in \mathbb{K}$ such that $F \subset \alpha D$. Thus, for $m \in H$, $s \in B$, $A \in K(X)$, we have that $\alpha^{-1}\chi_A s \in D$ and so $|m(A)s| \leq |\alpha|$. Therefore

$$\sup_{\phi \in \Phi, s \in B} |\phi(s)| \leq |\alpha|,$$

which proves that Φ is strongly bounded in E'. But then Φ is equicontinuous. Hence, there exists $p \in cs(E)$ such that

$$\Phi \subset \{s \in E : p(s) \leq 1\}^o.$$

Now

$$W = \{f \in G : \|f\|_{Y,p} \leq 1\} \subset H^o = D.$$

Indeed, let $\|f\|_{Y,p} \leq 1$ and let $V = \{x : p(f(x)) \leq 1\}$. For each clopen subset V_1 of V^c, we have that $m(V_1) = 0$ for all $m \in H$. For A a clopen subset of V and $x \in A$, we have $p(f(x)) \leq 1$ and so $|m(A)f(x)| \leq 1$, which implies that

$$\left|\int f\, dm\right| = \left|\int_V f\, dm\right| \leq 1.$$

Thus $W \subset D$ and the result follows.

Corollary 9.20 *$C_c(X)$ is polarly absolutely quasi-barrelled iff X is an aw_o-space.*

Corollary 9.21 *Assume that $E' \neq \{0\}$. If E is a bornological space and X an aw_o-space, then $C_c(X,E)$ is polarly absolutely quasi-barrelled. In particular this happens when E is metrizable.*

Definition 9.22 *A locally convex space E is said to be :*

1. *polarly $\aleph_o$-barrelled if every $w^\star$-bounded countable union of equicontinuous subsets of E' is equicontinuous.*
2. *polarly ℓ^∞-barrelled if every $w^\star$-bounded sequence in E' is equicontinuous.*
3. *polarly co-barrelled if every $w^\star$-null sequence in E' is equicontinuous.*

Theorem 9.23 *Assume that $E' \neq \{0\}$ and let $G = C_c(X,E)$. Consider the following conditions.*

1. *G is polarly $\aleph_o$-barrelled.*
2. *G is polarly ℓ^∞-barrelled .*
3. *G is polarly co-barrelled.*
4. *If a σ-compact subset A of X is bounding, then A is relatively compact.*

Then: (a. $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$.
(b). If E is a Fréchet space, then the four properties (1), (2), (3), (4) are equivalent.

Proof: Clearly $(1) \Rightarrow (2) \Rightarrow (3)$.
$(3) \Rightarrow (4)$. Let (Y_n) be a sequence of compact subsets of X, such that $A = \bigcup Y_n$ is bounding, and choose a non-zero element u of E'. Let p be defined on E by $p(s) = |u(s)|$. Then $\|u\|_p = 1$. By [22], p. 273, there exists $\mu_n \in M_\tau(X)$ with $N_{\mu_n}(x) = 1$ if $x \in Y_n$ and $N_{\mu_n}(x) = 0$ if $x \notin Y_n$. Let

$$m_n \in M(X,E'), \quad m_n(A) = \mu_n(A)u$$

for all $A \in K(X)$. Let $0 < |\lambda| < 1$. For each $f \in C(X,E)$, we have

$$\left|\int f\ dm_n\right| \leq \|f\|_{Y_n,p} \cdot \|m_n\|_p \leq \|f\|_{A,p}.$$

It follows that the sequence $H = (\lambda^n m_n)$ is $w^\star$-null and hence by (3) equicontinuous. Let Y be a compact subset of X and $q \in cs(E)$ be such that

$$\{f \in G : \|f\|_{Y,q} \leq 1\} \subset H^o.$$

But then $A \subset Y$ and so A is relatively compact. Finally, suppose that E is a Fréchet space and let (4) hold. Let (H_n) be a sequence of equicontinuous subsets of the dual space $M_c(X, E')$ of G such that $H = \bigcup H_n$ is $w^\star$-bounded. For each n, the set

$$Y_n = S(H_n) = \overline{\bigcup_{m \in H_n} supp(m)}$$

is compact. Also, the set

$$A = S(H) = \overline{\bigcup Y_n}$$

is bounding by [11], Propostion 6.6. By our hypothesis, A is compact. Since E is a Fréchet space, the space $F = (C_{rc}(X, E), \tau_u)$ is a Fréchet space whose dual can be identified with $M(X, E')$. As H is $\sigma(F', F)$-bounded, it follows that H is τ_u-equicontinuous. Thus, there exists $p \in cs(E)$ such that

$$\{f \in C_{rc}(X, E) : \|f\|_p \leq 1\} \subset H^o.$$

If $|\lambda| > 1$, then $\|m\|_p \leq |\lambda|$ for all $m \in H$. Now

$$\{f \in G : \|f\|_{A,p} \leq |\lambda^{-1}|\} \subset H^o.$$

This clearly completes the proof.

10 Tensor Products

Throughout this section, X, Y will be zero-dimensional Hausdorff topological spaces and E, F Hausdorff locally convex spaces. Let $B_{ou}(X)$ denote the collection of all $\phi \in \mathbb{K}^X$ for which $|\phi|$ is bounded, upper-semicontinuous and vanishes at infinity. For $\phi \in B_{ou}(X)$ and $p \in cs(E)$, let p_ϕ be the seminorm on $C_b(X, E)$ defined by

$$p_\phi(f) = \sup_{x \in X} p(\phi(x) f(x)).$$

As it is shown in [17], the topology β_o is generated by the family of seminorms

$$\{p_\phi : \phi \in B_{ou}(X),\ p \in cs(E)\}.$$

For $\phi_1, \phi_2 \in B_{ou}(X)$, it is proved in [7] that there exists $\phi \in B_{ou}(X)$ such that $|\phi| = \max\{|\phi_1|, |\phi_2|\}$. If $\phi_1 \in B_{ou}(X)$, $\phi_2 \in B_{ou}(Y)$, then the function

$$\phi = \phi_1 \times \phi_2 : X \times Y \to \mathbb{K},\ \phi(x, y) = \phi_1(x)\phi_2(y),$$

is in $B_{ou}(X \times Y)$ and, for each locally convex space G, the topology β_o on $C_b(X \times Y, G)$ is generated by the seminorms

$$p_{\phi_1 \times \phi_2}, \quad \phi_1 \in B_{ou}(X), \quad \phi_2 \in B_{ou}(Y), \quad p \in cs(G).$$

Let $E \otimes F$ be the tensor product of E, F equipped with the projective topology. For $f \in C_b(X, E)$, $g \in C_b(Y, F)$, define

$$f \odot g : X \times Y \to E \otimes F, \quad f \odot g(x, y) = f(x) \otimes g(y).$$

The bilinear map

$$\psi : E \times F \to E \otimes F, \quad \psi(a, b) = a \otimes b,$$

is continuous. Also the map $(x, y) \mapsto (f(x), g(x))$, from $X \times Y$ to $E \times F$, is continuous. Hence the composition $f \odot g$ is continuous. Since

$$p \otimes q(f \odot g(x, y)) = p(f(x) \cdot q(g(y)) \leq \|f\|_p \cdot \|g\|_q,$$

$f \odot g$ is also bounded.

Theorem 10.1 *The space G spanned by the functions*

$$(\chi_A s) \odot (\chi_B t),\ A \in K(X),\ B \in K(Y),\ s \in E,\ t \in F,$$

is β_o-dense in $C_b(X \times Y, E \otimes F)$.

Proof: Let $p \in cs(E)$, $q \in cs(F)$, $\phi_1 \in B_{ou}(X)$, $\phi_2 \in B_{ou}(Y)$, $\phi = \phi_1 \times \phi_2$. Consider the set

$$W = \{f \in C_b(X \times Y, E \otimes F) : (p \otimes q)_\phi(f) \leq 1\}$$

and let $f \in C_b(X \times Y, E \otimes F)$. We will finish the proof by showing that there exists $h \in G$ such that $f - h \in W$. To this end, we consider the set

$$D = \{(x, y) : |\phi_1(x)\phi_2(y)| \cdot p \otimes q(f(x, y)) \geq 1/2\}.$$

Then D is a compact subset of $X \times Y$. Let D_1, D_2 be the projections of D on X, Y, respectively. Then $D \subset D_1 \times D_2$. Choose $d > \|\phi_1\|, \|\phi_2\|$ and let $x \in D_1$. There exists a y such that $(x, y) \in D$ and so $\phi_1(x) \neq 0$. The set

$$Z_x = \{z \in X : |\phi_1(z)| < 2|\phi_1(x)|\}$$

is open and contains x. Using the compactness of D_2, we can find a clopen neighborhood W_x of x contained in Z_x such that $p \otimes q(f(z, y) - f(x, y)) < 1/d^2$ for all $z \in W_x$ and all $y \in D_2$. In view of the compactness of D_1, there are $x_1, x_2, \cdots, x_m \in D_1$ such that $D_1 \subset \bigcup_{k=1}^m W_{x_k}$. Let

$$A_1 = W_{x_1}, \quad A_{k+1} = W_{x_{k+1}} \setminus \bigcup_{j=1}^{k} W_{x_j}, \quad k = 1, 2, \ldots, m-1.$$

Keeping those of the A_i which are not empty, we may assume that $A_k \neq \emptyset$ for all $1 \leq k \leq m$. For $k = 1, \ldots, m$, there are pairwise disjoint clopen subsets $B_{k,1}, \ldots, B_{k,n_k}$ of Y covering D_2 and $y_{kj} \in B_{k,j}$ such that

$$p \otimes q(f(x_k, y) - f(x_k, y_{kj})) < 1/d^2$$

if $y \in B_{k,j}$. Let

$$h = \sum_{k=1}^{m} \sum_{j=1}^{n_k} \chi_{A_k} \times \chi_{B_{k,j}} \cdot f(x_k, y_{kj}).$$

Then $h \in G$. We will prove that

$$|\phi_1(x)\phi_2(y)| \cdot p \otimes q(f(x,y) - h(x,y)) \leq 1$$

for all $x \in X, y \in Y$. To see this, we consider the three possible cases.
Case I. $x \notin \bigcup_{k=1}^{m} A_k$. Then $h(x,y) = 0$. Also $(x,y) \notin D$ and thus

$$|\phi_1(x)\phi_2(y) \cdot p \otimes q(f(x,y)) \leq 1/2.$$

Case II. $x \in A_k$, $y \in D_2$. There exists j such that $y \in B_{k,j}$. Now

$$p \otimes q(f(x,y) - f(x_k,y)) < 1/d^2 \quad \textit{and} \quad p \otimes q(f(x_k,y) - f(x_k,y_{kj})) \leq 1/d^2.$$

Since $h(x,y) = f(x_k, y_{kj})$, we have

$$|\phi_1(x)\phi_2(y)| \cdot p \otimes q(f(x,y) - h(x,y)) \leq 1.$$

Case III. $x \in A_k, y \notin D_2$. Then $(x,y) \notin D$ and so $|\phi_1(x)\phi_2(y)| \cdot p \otimes q(f(x,y)) < 1/2$. If $h(x,y) \neq 0$, then $y \in B_{k,j}$, for some j, and so $h(x,y) = f(x_k, y_{kj})$ and $p \otimes q(f(x_k,y) - f(x_k,y_{kj})) < 1/d^2$. Since $x \in W_{x_k}$, we have $|\phi_1(x)| < 2|\phi_1(x_k)|$. Thus

$$|\phi_1(x)\phi_2(y)| \cdot p \otimes q(f(x_k,y)) \leq 2|\phi_1(x_k)\phi_2(y)| \cdot p \otimes q(f(x_k,y)) \leq 1$$

since $(x_k, y) \notin D$. It follows that

$$|\phi_1(x)\phi_2(y)| \cdot p \otimes q(f(x,y) - h(x,y)) \leq 1.$$

Thus $f - h \in W$, which completes the proof.

Lemma 10.2 *Let $p \in cs(E)$, $q \in cs(F)$ and $u \in E \otimes F$. Then :*

1. *If $u = \sum_{i=1}^{n} x_i \otimes y_i = \sum_{j=1}^{m} a_j \otimes b_j$, then for all $x' \in E'$, we have*

$$\sum_{i=1}^{n} x'(x_i) y_i = \sum_{j=1}^{m} x'(a_j) b_j.$$

2. *If p is polar, then, for any $u = \sum_{i=1}^{n} x_i \otimes y_i$, we have*

$$p \otimes q(u) = \sup\{q(\sum_{i=1}^{n} x'(x_i) y_i) : x' \in E', |x'| \leq p\}.$$

Proof: (1). Let $h \in F^*$ and consider the bilinear map

$$\omega : E \times F \to \mathbb{K}, \quad \omega(x,y) = x'(x)h(y).$$

Let $\hat{\omega} : E \otimes F \to \mathbb{K}$ be the corresponding linear map. Then

$$\sum_{i=1}^{n} x'(x_i)h(y_i) = \hat{\omega}\left(\sum_{i=1}^{n} x_i \otimes y_i\right) = \hat{\omega}\left(\sum_{j=1}^{m} a_j \otimes b_j\right) = \sum_{j=1}^{m} x'(a_j)h(b_j).$$

Since this holds for all $h \in F^*$, (1) follows.
(2). Let $d = \sup_{|x'|\le p} q(\sum_{i=1}^{n} x'(x_i)y_i)$. For any representation $u = \sum_{j=1}^{m} a_j \otimes b_j$ of u and any $x' \in E'$, with $|x'| \le p$, we have

$$q\left(\sum_{j=1}^{m} x'(a_j)b_j\right) \le \sup_j |x'(a_j)|q(b_j) \le \sup_j p(a_j)q(b_j)$$

and so $d \le sup_j p(a_j)q(b_j)$, which proves that $d \le p \otimes q(u)$. On the other hand, let $u = \sum_{i=1}^{n} x_i \otimes y_i$ and let G be the space spanned by the the set $\{y_1, \dots, y_n\}$. Given $0 < t < 1$, there exists a basis $\{w_1, \dots, w_m\}$ of G which is t-orthogonal with respect to the seminorm q. We may write u in the form $u = \sum_{k=1}^{m} z_k \otimes w_k$. For $x' \in E', |x'| \le p$, we have

$$q\left(\sum_{k=1}^{m} x'(z_k)w_k\right) \ge t \cdot \max_{1\le k\le m} |x'(z_k)|q(w_k),$$

and so

$$\begin{aligned} \sup_{|x'|\le p} q\left(\sum_{k=1}^{m} x'(z_k)w_k\right) &\ge t \cdot \sup_{|x'|\le p} \max_k |x'(z_k)|q(w_k) \\ &= t \cdot \max_k \left[\sup_{|x'|\le p} |x'(z_k)|\right] q(w_k) \\ &= t \cdot \max_k p(z_k)q(w_k) \ge t \cdot p \otimes q(u). \end{aligned}$$

Since $0 < t < 1$ was arbitrary, we get that $d \ge p \otimes q(u)$ and so $d = p \otimes q(u)$.

Lemma 10.3 *If $p \in cs(E)$ is polar and $\phi \in B_{ou}(X)$, then p_ϕ is a polar continuous seminorm on $(C_b(X,E), \beta_o)$.*

Proof Let $p_\phi(f) > \theta > 0$. There exists $x \in X$ such that $|\phi(x)|p(f(x)) > \theta$ and so $p(f(x)) > \alpha = \theta/|\phi(x)|$. Since p is polar, there exists $x' \in E', |x'| \le p$, such that $|x'(f(x))| > \alpha$. Let

$$v : C_b(X,E) \to \mathbb{K}, \quad v(g) = \phi(x)x'(g(x)).$$

Then v is linear and $|v| \le p_\phi$. Moreover, $|v(f)| > \theta$, which proves that p_ϕ is polar.

Theorem 10.4 *If E is polar, then there exists a linear homeomorphism*

$$\omega : (C_b(X,E),\, \beta_o) \otimes (C_b(Y,F),\, \beta_o) \to (C_b(X \times Y, E \otimes F),\, \beta_o)$$

onto a β_o-dense subspace of $C_b(X \times Y,\, E \otimes F)$. Moreover $\omega(f \otimes g) = f \odot g$ for all $f \in C_b(X,E)$, $g \in C_b(Y,F)$.

Proof: Let

$$G = (C_b(X,E),\, \beta_o) \otimes (C_b(Y,F),\, \beta_o).$$

The bilinear map

$$T : (C_b(X,E),\, \beta_o) \times (C_b(Y,F),\, \beta_o) \to (C_b(X \times Y, E \otimes F),\, \beta_o),$$

$T(f,g) = f \odot g$, is continuous. Indeed, let $p \in cs(E)$ be polar, $q \in cs(F)$, $\phi_1 \in B_{ou}(X)$, $\phi_2 \in B_{ou}(Y)$, $\phi = \phi_1 \times \phi_2$. Then

$$\begin{aligned}
(p \otimes q)_\phi(f \odot g) &= \sup_{x,y} |\phi_1(x)\phi_2(y)| p \otimes q((f(x) \otimes g(y)) \\
&= \sup_{x,y} |\phi(x,y)| p(f(x)) q(g(y)) = p_{\phi_1}(f) q_{\phi_2}(g),
\end{aligned}$$

and hence T is continuous. Let

$$\omega : G \to (C_b(X \times Y,\, E \otimes F),\, \beta_o)$$

be the corresponding continuous linear map.
Claim. For each $u \in G$, we have

$$(p \otimes q)_\phi(\omega(u)) = p_{\phi_1} \otimes q_{\phi_2}(u).$$

Indeed, if $u = \sum_{k=1}^n f_k \otimes g_k$, then

$$\begin{aligned}
|\phi_1(x)\phi_2(y)| \cdot p \otimes q(\omega(u)(x,y)) &= |\phi_1(x)\phi_2(y)| \cdot p \otimes q \left(\sum_{k=1}^n f_k(x) \otimes g_k(y) \right) \\
&\le |\phi_1(x)\phi_2(y)| \cdot \max_k p(f_k(x)) q(g_k(y)) \\
&\le \max_k p_{\phi_1}(f_k) q_{\phi_2}(g_k).
\end{aligned}$$

Thus

$$(p \otimes q)_\phi(\omega(u)) \le \max_k p_{\phi_1}(f_k) q_{\phi_2}(g_k),$$

which proves that

$$(p \otimes q)_\phi(\omega(u)) \le p_{\phi_1} \otimes q_{\phi_2}(u).$$

On the other hand, given $0 < t < 1$, there exists a representation $u = \sum_{k=1}^n f_k \otimes g_k$ of u such that the set $\{g_1, \ldots, g_n\}$ is t-orthogonal with respect to the seminorm q_{ϕ_2}. Now

$$\begin{aligned}
(p \otimes q)_\phi(\omega(u)) &= \sup_{x,y} |\phi_1(x)\phi_2(y)| p \otimes q \left(\sum_{k=1}^n f_k(x) g_k(y) \right) \\
&= \sup_{x,y} \left[|\phi_1(x)\phi_2(y)| \cdot \sup\{ q \left(\sum_{k=1}^n x'(f_k(x)) g_k(y) \right) : |x'| \le p \} \right] \\
&= \sup_{x} \left[|\phi_1(x)| \cdot \sup_{|x'| \le p} \{ \sup_{y} |\phi_2(y)| \cdot q(\sum_{k=1}^n x'(f_k(x)) g_k(y)) \} \right] \\
&= \sup_{x} \left[|\phi_1(x)| \cdot \sup_{|x'| \le p} q_{\phi_2} \left(\sum_{k=1}^n x'(f_k(x)) g_k \right) \right] \\
&\ge t \cdot \sup_{x} \left[|\phi_1(x)| \cdot \sup_{|x'| \le p} \max_k |x'(f_k(x))| \cdot q_{\phi_2}(g_k) \right] \\
&= t \cdot \sup_{x} \left[|\phi_1(x)| \cdot \left(\max_k p(f_k(x)) q_{\phi_2}(g_k) \right) \right] \\
&= t \cdot \max_k p_{\phi_1}(f_k) q_{\phi_2}(g_k) \ge t \cdot p_{\phi_1} \otimes q_{\phi_2}(u).
\end{aligned}$$

Since $0 < t < 1$ was arbitrary, we get that $(p \otimes q)_\phi(\omega(u)) \ge p_{\phi_1} \otimes q_{\phi_2}(u)$ and the claim follows.
It is now clear that ω is one-to-one and, for $M = \omega(G)$, the map $\omega : G \to (M, \beta_o)$ is a homeomophism. Since, for $A \in K(X)$, $B \in K(Y)$, $a \in E$, $b \in F$, we have that $(\chi_A a) \odot (\chi_B b) \in M$, it follows that M is β_o-dense in $(C_b(X \times Y, E \otimes F), \beta_o)$ in view of Theorem 10.1. This completes the proof.

For $x' \in E'$ and $y' \in F'$, we denote by $x' \otimes y'$ the unique element of $(E \otimes F)'$ defined by

$$x' \otimes y'(s_1 \otimes s_2) = x'(s_1) y'(s_2).$$

Theorem 10.5 *Assume that E is polar and let $m_1 \in M_t(X, E')$, $m_2 \in M_t(Y, F')$. Then there exists a unique $\bar{m} \in M_t(X \times Y, (E \otimes F)')$ such that*

$$\bar{m}(A \times B) = m_1(A) \otimes m_2(B)$$

for $A \in K(X)$, $B \in K(Y)$. Moreover, for $g \in C_b(X, E), f \in C_b(Y, F)$, $h = g \odot f$, we have

$$\int h \, d\bar{m} = \left(\int g \, dm_1 \right) \cdot \left(\int f \, dm_2 \right).$$

Proof: Since m_1 is β_o-continuous on $C_b(X, E)$, there exist $\phi_1 \in B_{ou}(X)$ and a polar continuous seminorm p on E such that $|\int g \, dm_1| \le p_{\phi_1}(g)$ for all $g \in C_b(X, E)$. Similarly, there exist $\phi_2 \in B_{ou}(Y)$ and $q \in cs(F)$ such that $|\int f \, dm_2| \le q_{\phi_2}(f)$ for all $f \in C_b(Y, F)$. Consider the bilinear map

$$T : (C_b(X, E), \beta_o) \times (C_b(Y, F), \beta_o) \to \mathbb{K}, \quad T(g, f) = \left(\int g \, dm_1 \right) \cdot \left(\int f \, dm_2 \right).$$

Then T is continuous since $|T(g,f)| \leq p_{\phi_1}(g) \cdot q_{\phi_2}(f)$. Hence the corresponding linear map

$$\psi : G = (C_b(X,E), \beta_o) \otimes (C_b(Y,F), \beta_o) \to \mathbb{K}$$

is continuous. Let ω be as in the preceding Theorem and $M = \omega(G)$. The linear map

$$v : (M, \beta_o) \to \mathbb{K}, \quad v = \psi \circ \omega^{-1},$$

is continuous. Since M is β_o-dense in $C_b(X \times Y, E \otimes F)$, there exists a unique β_o-continuous linear extension $\tilde{v}$ of v to all of $C_b(X \times Y, E \otimes F)$. Let

$$\bar{m} \in M_t(X \times Y, (E \otimes F)')$$

be such that $\tilde{v}(h) = \int h\, d\bar{m}$ for all $h \in C_b(X \times Y, E \otimes F)$. Taking

$$h = (\chi_A s_1) \odot (\chi_B s_2) = \chi_{A \times B} s_1 \otimes s_2,$$

where $A \in K(X)$, $B \in K(Y)$, $s_1 \in E$, $s_2 \in F$, we get that

$$\begin{aligned} \bar{m}(A \times B)(s_1 \otimes s_2) &= \int h\, d\bar{m} = \psi((\chi_A s_1) \otimes (\chi_B s_2)) \\ &= (m_1(A)s_1) \otimes (m_2(B)s_2)) = [m_1(A) \otimes m_2(B)](s_1 \otimes s_2). \end{aligned}$$

Thus $\bar{m}(A \times B) = m_1(A) \otimes m_2(B)$. If $g \in M_b(X,E)$, $f \in M_b(Y,F)$ and $h = g \odot f$, then

$$\int h\, d\bar{m} = \tilde{v}(h) = \psi(g \otimes f) = \left(\int g\, dm_1\right) \cdot \left(\int f\, dm_2\right).$$

Finally, let $\mu \in M_t(X \times Y, (E \otimes F)')$ be such that $\mu(A \times B) = m_1(A) \otimes m_2(B)$ for all $A \in K(X)$, $B \in K(Y)$. The map

$$v_1 : C_b(X \times Y, E \otimes F) \to \mathbb{K}, \quad v_1(h) = \int h\, d\mu,$$

is β_o-continuous. Taking

$$h = (\chi_A s_1) \odot (\chi_B s_2) = \chi_{A \times B} s_1 \otimes s_2),$$

where $A \in K(X)$, $B \in K(Y)$, $s_1 \in E$, $s_2 \in F$, we have that $v_1(h) = \tilde{v}(h)$. In view of Theorem 10.1, we see that $v_1 = \tilde{v}$ on a β_o-dense subspace of $C_b(X \times Y, E \otimes F)$ and hence $v_1 = \tilde{v}$, which implies that $\bar{m} = \mu$. This completes the proof.

Definition 10.6 *If $m_1, m_2, \bar{m}$ are as in the preceding Theorem, we will call $\bar{m}$ the tensor product of m_1, m_2 and denote it by $m_1 \otimes m_2$.*

Theorem 10.7 *Assume that E is polar and let $m_1 \in M_{t,p}(X, E')$, $m_2 \in M_{t,q}(Y, F')$. Suppose that p is polar. Then*

1. *$\bar{m} = m_1 \otimes m_2 \in M_{t,p \otimes q}(X \times Y, (E \otimes F)')$ and $\|\bar{m}\|_{p \otimes q} = \|m_1\|_p \|m_2\|_q$.*

2. *If* $\phi_1 \in B_{ou}(X)$, $\phi_2 \in B_{ou}(Y)$ *are such that* $|\int g\, dm_1| \le p_{\phi_1}(g)$, *for all* $g \in C_b(X, E)$, *and* $|\int f\, dm_2| \le p_{\phi_2}(f)$, *for all* $f \in C_b(Y, F)$, *then for* $\phi = \phi_1 \times \phi_2$, *we have*

$$\left|\int h\, d\bar{m}\right| \le (p \otimes q)_\phi(h), \quad \text{for all} \quad h \in C_b(X \times Y, E \otimes F).$$

Proof: Let ϕ_1 and ϕ_2 be as in the Theorem. For $g \in C_b(X, E)$, $f \in C_b(Y, F)$ and $h = g \odot f$, we have

$$\left|\int h\, d\bar{m}\right| = \left|\left(\int g\, dm_1\right) \cdot \left(\int f\, dm_2\right)\right| \le p_{\phi_1}(g) q_{\phi_2}(f).$$

It is easy to see that $\|\phi h\|_{p\otimes q} = \|\phi_1 g\|_p \cdot \|\phi_2 f\|_q$. Thus

$$\left|\int h\, d\bar{m}\right| \le \|\phi h\|_{p\otimes q}.$$

Since both maps $h \mapsto (p \otimes q)_\phi(h)$ and $h \mapsto \int h \bar{m}$ are β_o-contiuous and M is β_o-dense, it follows that

$$\left|\int h\, d\bar{m}\right| \le \|\phi h\|_{p\otimes q}.$$

for all $h \in C_b(X \times Y, E \otimes F)$. Hence $\bar{m} \in M_{t,p\otimes q}(X \times Y, (E \otimes F)')$. For $g \in C_b(X, E)$, $f \in C_b(Y, F)$, $h = g \odot f$, we have

$$\begin{aligned} \left|\int h\, d\bar{m}\right| = \left|\left(\int g\, dm_1\right) \cdot \left(\int f\, dm_2\right)\right| &\le \|m_1\|_p \cdot \|g\|_p \cdot \|m_2\|_q \cdot \|f\|_q \\ &= [\|m_1\|_p \cdot \|m_2\|_q] \cdot [\|h\|_{p\otimes q}]. \end{aligned}$$

Thus $\|\bar{m}\|_{p\otimes q} \le \|m_1\|_p \cdot \|m_2\|_q = d$. If $d > 0$ and $0 < \epsilon_1 < \|m_1\|_p$, $0 < \epsilon_2 < \|m_2\|_q$, then there are $A \in K(X)$, $B \in K(Y)$, $s_1 \in E$, $s_2 \in F$, such that

$$\frac{|m_1(A)s_1|}{p(s_1)} > \|m_1\|_p - \epsilon_1, \quad \frac{|m_2(B)s_2|}{q(s_2)} > \|m_2\|_q - \epsilon_2.$$

Now

$$\|\bar{m}\|_{p\otimes q} \ge \frac{|\bar{m}(A \times B)s_1 \otimes s_2|}{p \otimes q(s_1 \otimes s_2)} > (\|m_1\|_p - \epsilon_1) \cdot (\|m_2\|_q - \epsilon_2).$$

Taking $\epsilon_1 \to 0$, $\epsilon_2 \to 0$, we get $\|\bar{m}\|_{p\otimes q} \ge \|m_1\|_p \cdot \|m_2\|_q$, which completes the proof.

Lemma 10.8 *Let* $m \in M_p(X, E')$, $V \in K(X)$ *and*

$$\alpha = \sup\{|m(A)s| : A \in K(X), A \subset V, p(s) \le 1\}.$$

Then

1. *for any* $\lambda \in \mathbb{K}$, *with* $|\lambda| > 1$, *we have* $\alpha \le m_p(V) \le |\lambda|\alpha$.

2. *If the valuation of* $\mathbb{K}$ *is dense or if it is discrete and* $p(E) \subset |\mathbb{K}|$, *then* $m_p(V) = \alpha$.

Proof: (1). If $p(s) \leq 1$ and $A \in K(X)$, $A \subset V$, then $|m(A)s)| \leq m_p(V) \cdot p(s) \leq m_p(V)$ and so $\alpha \leq m_p(V)$.On the other hand, if $p(s) > 0$, then there exists $\gamma \in \mathbb{K}$ with $|\gamma| \leq p(s) \leq |\gamma\lambda|$. Now, for $A \subset V$, we have

$$\alpha \geq |m(A)(\gamma^{-1}\lambda^{-1}s)| \geq |\lambda^{-1}| \cdot \frac{|m(A)s|}{p(s)}.$$

It follows that $\alpha|\lambda| \geq m_p(V)$.
(2). It is clear from (1) that $\alpha = m_p(V)$ if the valuation is dense. Suppose that the valuation is discrete and $p(E) \subset |\mathbb{K}|$. If $p(s) > 0$, then there exists $\gamma \in \mathbb{K}$, with $p(s) = |\gamma|$. For $A \subset V$, we have $\frac{|m(A)s|}{p(s)} = |m(A)(\gamma^{-1}s)| \leq \alpha$ and so $m_p(V) \leq \alpha$, which completes the proof.

Theorem 10.9 *Assume that E is polar and let $p \in cs(E)$ be polar, $q \in cs(F)$. If $m_1 \in M_{t,p}(X, E')$, $m_2 \in M_{t,q}(Y, F')$ and $\bar{m} = m_1 \otimes m_2$, then, for $|\lambda| > 1$, we have*

$$N_{m_1,p}(x) \cdot N_{m_2,q}(y) \leq N_{\bar{m},p\otimes q}(x, y) \leq |\lambda| N_{m_1,p}(x) \cdot N_{m_2,q}(y).$$

If the valuation of $\mathbb{K}$ is dense or if it is discrete and $q(F) \subset |\mathbb{K}|$, then

$$N_{m_1,p}(x) \cdot N_{m_2,q}(y) = N_{\bar{m},p\otimes q}(x, y)$$

Proof: Let Z be a clopen neighborhood of (x, y). There are $A \in K(X)$, $B \in K(Y)$ such that $(x, y) \in A \times B \subset Z$. For $s_1 \in E$, $s_2 \in F$, $s = s_1 \otimes s_2$, with $p(s_1) \leq 1$, $q(s_2) \leq 1$, we have

$$\sup_{A_1 \subset A, B_1 \subset B} \frac{|m_1(A_1)s_1| \cdot |m_2(B_1)s_2|}{p \otimes q(s)} \leq |\bar{m}|_{p\otimes q}(Z)$$

and so

$$N_{m_1,p}(x) \cdot N_{m_2,q}(y) \leq |m_1|_p(A) \cdot |m_2|_q(B) \leq |\bar{m}|_{p\otimes q}(Z).$$

Hence

$$N_{m_1,p}(x) \cdot N_{m_2,q}(y) \leq N_{\bar{m},p\otimes q}(x, y).$$

On the other hand, let $N_{m_1,p}(x) \cdot N_{m_2,q}(y) < \theta$. There are clopen sets V_1, V_2, $x \in V_1$, $y \in V_2$, $|m_1|_p(V_1) \cdot |m_2|_q(V_2) < \theta$. Let

$$d = \sup\{|\bar{m}(D)u| : D \subset V_1 \times V_2,\, p \otimes q(u) \leq 1\}.$$

Let $u \in E \otimes F$ with $p \otimes q(u) \leq 1$. Given $0 < t < 1$, there exists a representation $u = \sum_{j=1}^{N} s_j \otimes a_j$ of u such that the set $\{a_1, \ldots, a_N\}$ is t-orthogonal with respect to the seminorm q. Now

$$\begin{aligned}
1 \geq p \otimes q(u) &= \sup_{|x'| \leq p} q\left(\sum_{j=1}^{N} x'(s_j)a_j\right) \\
&\geq t \cdot \sup_{|x'| \leq p} \max_j |x'(s_j)| q(a_j) \\
&= t \cdot \max_j p(s_j) q(a_j).
\end{aligned}$$

Let $0 < \epsilon < \theta$. There exists a compact subset G of $X \times Y$ such that $|\bar{m}|_{p\otimes q}(W) < \epsilon$ if the clopen set W is disjoint from G. Let D be a clopen subset of $V_1 \times V_2$. For each $z = (a, b) \in G \cap D$, there are clopen neighborhoods W_z, M_z of a, b, respectively, with $(a, b) \in W_z \times M_z \subset D$.

In view of the compactness of $G \cap D$, there are $z_i = (x_i, y_i) \in G \cap D$, $i = 1, \ldots, n$, such that

$$G \cap D \subset D_1 = \bigcup_{i=1}^{n} W_{z_i} \times M_{z_i} \subset D.$$

There are pairwise disjoint clopen rectangles $A_j \times B_j$, $j = 1, \ldots, k$, such that

$$D_1 = \bigcup_{j=1}^{k} A_j \times B_j.$$

Now

$$\bar{m}(D)s_i \otimes a_i = \bar{m}(D \setminus D_1)s_i \otimes a_i + \sum_{j=1}^{k} \bar{m}(A_j \times B_j)s_i \otimes a_i.$$

Since $D \setminus D_1$ is disjoint from G, it follows that

$$|\bar{m}(D \setminus D_1)s_i \otimes a_i| \leq |\bar{m}|_{p\otimes q}(D \setminus D_1) \cdot p \otimes q(s_i \otimes a_i) \leq \epsilon/t < \theta/t.$$

Also,

$$\begin{aligned} |\bar{m}(A_j \times B_j)s_i \otimes a_i| &= |m_1(A_j)s_i| \cdot |m_2(B_j)a_i| \\ &\leq |m_1|_p(V_1)p(s_i) \cdot |m_2|_q(V_2)q(a_i) \\ &\leq \frac{|m_1|_p(V_1) \cdot |m_2|_q(V_2)}{t} < \theta/t. \end{aligned}$$

Thus $|\bar{m}(D)s_i \otimes a_i| < \theta/t$ and hence

$$|\bar{m}(D)u| \leq \max_i |\bar{m}(D)s_i \otimes a_i| < \theta/t.$$

This proves that $d \leq \theta/t$ and so $|\bar{m}|_{p\otimes q}(V_1 \times V_2) \leq |\lambda| \cdot \theta/t$, which shows that $N_{\bar{m},p\otimes q}(x, y) \leq |\lambda|\theta/t$. Therefore

$$N_{\bar{m},p\otimes q}(x, y) \leq \frac{|\lambda|}{t} \cdot N_{m_1,p}(x)N_{m_2,q}(y).$$

Since $0 < t < 1$ was arbitrary, we get that

$$N_{\bar{m},p\otimes q}(x, y) \leq |\lambda| \cdot N_{m_1,p}(x)N_{m_2,q}(y).$$

If the valuation of $\mathbb{K}$ is dense or if it is discrete and $q(F) \subset |\mathbb{K}|$, then

$$d = |\bar{m}|_{p\otimes q}(V_1 \times V_2) \leq \theta/t$$

and hence $N_{\bar{m},p\otimes q}(x, y) \leq \theta/t$. Since $0 < t < 1$ was arbitrary, we have that $N_{\bar{m},p\otimes q}(x, y) \leq \theta$, which shows that

$$N_{\bar{m},p\otimes q}(x) \leq N_{m_1,p}(x) \cdot N_{m_2,q}(y),$$

and the result follows.

Note 10.10 *If p is polar and $q(F) \subset |\mathbb{K}|$, then $p \otimes q(E \otimes F) \subset |\mathbb{K}|$.*

This follows from the fact that, for $u = \sum_{i=1}^{n} x_i \otimes y_i$, we have

$$p \otimes q(u) = \sup_{|x'| \leq p} q\left(\sum_{i=1}^{n} x'(x_i) y_i\right).$$

We have the following easily established

Theorem 10.11 *Let m_1, m_2, $\bar{m}$ be as in Theorem 10.9. If $V_1 \in K(X)$, $V_2 \in K(Y)$ and $|\lambda| > 1$, then*

$$|m_1|_p(V_1) \cdot |m_2|_q(V_2) \leq |\bar{m}|_{p\otimes q}(V_1 \times V_2) \leq |\lambda| \cdot |m_1|_p(V_1) \cdot |m_2|_q(V_2).$$

If the valuation of $\mathbb{K}$ is dense or if it is discrete and $q(F) \subset |\mathbb{K}|$, then

$$|m_1|_p(V_1) \cdot |m_2|_q(V_2) = |\bar{m}|_{p\otimes q}(V_1 \times V_2).$$

Theorem 10.12 *Let m_1, m_2, $\bar{m}$ be as in Theorem 10.9. Then*

$$supp(\bar{m}) = supp(m_1) \times supp(m_2).$$

Proof: Let $A_1 = \{x \in X : N_{m_1,p}(x) \neq 0\}$, $A_2 = \{y \in Y : N_{m_2,q}(y) \neq 0\}$, and $A = \{(x,y) : N_{\bar{m},p\otimes q}(x,y) \neq 0\}$. Then $A = A_1 \times A_2$. The result now follows from Theorem 2.1.

Theorem 10.13 *Assume that E is polar and let $p \in cs(E)$ be polar and $q \in cs(F)$. Let $m_1 \in M_{t,p}(X, E')$, $m_2 \in M_{t,q}(Y, F')$ and $\bar{m} = m_1 \otimes m_2$. If $g \in E^X$ is Q-integrable with respect to m_1, $f \in F^Y$ is Q-integrable with respect to m_2 and $h = g \odot f$, then :*

1. $Q_{\bar{m},h}(x,y) = Q_{m_1,g}(x) \cdot Q_{m_2,f}(y)$.
2. *h is Q-integrable with respect to $\bar{m}$ and*

$$(Q)\int h\, d\bar{m} = \left[(Q)\int g\, dm_1\right] \cdot \left[(Q)\int f\, dm_2\right].$$

Proof: Let $V_1 \in K(X)$, $V_2 \in K(Y)$, $x \in V_1$, $y \in V_2$. Then
$\sup\{|\bar{m}(D)h(x,y)| : D \in K(X \times Y), D \subset V_1 \times V_2\}$
$\geq \sup_{A\in K(X), A\subset V_1} \sup_{B\in K(Y), B\subset V_2} |m_1(A)g(x)| \cdot |m_2(B)f(y)|$
$\geq Q_{m_1,g}(x) \cdot Q_{m_2,f}(y)$.
It follows that

$$Q_{\bar{m},h}(x,y) \geq Q_{m_1,g}(x) \cdot Q_{m_2,f}(y).$$

On the other hand, let $\epsilon > 0$. There are clopen neighborhoods V_1 and V_2 of x, y, respectively, such that

$$\sup_{A\in K(X), A\subset V_1} |m_1(A)g(x)| < Q_{m_1,g}(x) + \epsilon$$

and

$$\sup_{B\in K(Y), B\subset V_2} |m_2(B)f(y)| < Q_{m_2,f}(y) + \epsilon.$$

Let now $G \in K(X \times Y)$ be contained in $V_1 \times V_2$ and let $d_1 > 0$ be such that $d_1 \cdot p(g(x))q(f(y)) < \epsilon$. There exists a compact subset D of $X \times Y$ such that $|\bar{m}|(O) < d_1$ if the clopen set O is disjoint from D. For each $z = (a, b) \in G \cap D$, there are clopen sets s W_z, M_z such that $z \in W_z \times M_z \subset G$. By the compactness of $D \cap G$, there are $z_k = (a_k, b_k) \in D \cap G$ such that

$$D \cap G \subset O = \bigcup_{k=1}^{n} W_{z_k} \times M_{z_k}.$$

There are pairwise disjoint clopen rectangles $A_i \times B_i$ such that $O = \bigcup_{i=1}^{N} A_i \times B_i$. Since $G \setminus O$ is disjoint from D, we have

$$|\bar{m}(G \setminus O)g(x) \otimes f(y)| \le |\bar{m}|_{p\otimes q}(G \setminus O)p(g(x))q(f(y)) < \epsilon.$$

Also,

$$|\bar{m}(A_i \times B_i)g(x) \otimes f(y)| = |m_1(A_i)g(x)| \cdot |m_2(B_i)f(y)| \le [Q_{m_1,g}(x) + \epsilon] \cdot [Q_{m_2,f}(y) + \epsilon]$$

since $A_i \subset V_1$, $B_i \subset V_2$. Thus

$$|\bar{m}(O)g(x) \otimes f(y)| \le [Q_{m_1,g}(x) + \epsilon] \cdot [Q_{m_2,f}(y) + \epsilon]$$

and so

$$|\bar{m}(G)h(x, y)| \le \max\{\epsilon, \quad [Q_{m_1,g}(x) + \epsilon] \cdot [Q_{m_2,f}(y) + \epsilon]\}.$$

Therefore

$$Q_{\bar{m},h}(x, y) \le \max\{\epsilon, \quad [Q_{m_1,g}(x) + \epsilon] \cdot [Q_{m_2,f}(y) + \epsilon]\}.$$

Taking $\epsilon \to 0$, we get that

$$Q_{\bar{m},h}(x, y) \le Q_{m_1,g}(x) \cdot Q_{m_2,f}(y)$$

which completes the proof of (1).

(2). Let $0 < \epsilon < 1$ and choose $0 < d < \epsilon$ such that $d \cdot \|g\|_{Q_{m_1}} < \epsilon, \quad d \cdot \|f\|_{Q_{m_2}} < \epsilon$. There are $g_1 \in S(X, E)$, $f_1 \in S(Y, F)$ such that

$$\|g - g_1\|_{Q_{m_1}} < d, \quad \|f - f_1\|_{Q_{m_2}} < d.$$

Let $h_1 = g_1 \odot f_1 \in S(X \times Y, E \otimes F)$. Then

$$h_1(x, y) - h(x, y) = [g_1(x) - g(x)] \otimes [f_1(y) - f(y)] + g(x) \otimes [f_1(y) - f(y)] + [g_1(x) - g(x)] \otimes f(y).$$

Using (1), we get

$$Q_{\bar{m},\, h_1 - h}(x, y) \le \max\{d^2,\ d \cdot \|g\|_{Q_{m_1}},\ d \cdot \|f\|_{Q_{m_2}}\},$$

and thus $\|h_1 - h\|_{Q_{\bar{m}}} \leq \epsilon$, which proves that h is Q-integrable with respect to $\bar{m}$. Finally, let $(g_n) \subset S(X, E)$, $(f_n) \subset S(Y, F)$ be such that

$$\|g - g_n\|_{Q_{m_1}} \to 0, \quad \|f - f_n\|_{Q_{m_2}} \to 0.$$

If $h_n = g_n \odot f_n \in S(X\times, E \otimes F)$, then $\|h - h_n\|_{Q_{\bar{m}}} \to 0$ and so

$$(Q)\int h\, d\bar{m} = \lim \int h_n\, d\bar{m}, \quad (Q)\int g\, dm_1 = \lim \int g_n\, dm_1,$$

and

$$(Q)\int f\, dm_2 = \lim \int f_n\, dm_2.$$

Since

$$\int h_n\, d\bar{m} = \left(\int g_n\, dm_1\right) \cdot \left(\int f_n\, dm_2\right),$$

the result follows.

References

[1] J. Aguayo, A. K. Katsaras and S. Navarro, On the dual space for the strict topology β_1 and the space M(X) in function spaces, Cont. Math. vol. **384** (2005), 15-37.

[2] J. Aguayo, N. de Grande-de Kimpe and S. Navarro, Strict locally convex topologies on $BC(X, \mathbb{K})$, Lecture Notes in Pure and Applied Mathematics, v. **192**, Marcel Dekker, New York (1997), 1-9.

[3] J. Aguayo, N. de Grande-de Kimpe and S. Navarro, Zero-dimensional pseudocompact and ultraparacopmpact spaces, Lecture Notes in Pure and Applied Mathematics, v. **192**, Marcel Dekker, New York (1997),11-37.

[4] J. Aguayo, N. de Grande-de Kimpe and S. Navarro, Strict topologies and duals in spaces of functions, Lecture Notes in Pure and Applied Mathematics, v. **207**, Marcel Dekker, New York (1999), 1-10.

[5] G. Bachman, E. Beckenstein, L. Narici and S. Warner, Rings of continuous functions with values in a topological field, Trans. Amer. Math. Soc. **204** (1975), 91-112.

[6] N. de Grande-de Kimpe and S. Navarro, Non-Archimedean nuclearity and spaces of continuous functions, Indag. Math., N.S. **2(2)**(1991), 201-206.

[7] W. Govaerts, Locally convex spaces of non-Archimedean valued functions, Pacific J. of Math., vol. **109**, no 2 (1983), 399-410.

[8] A. K. Katsaras, Duals of non-Archimedean vector-valued function spaces, Bull. Greek Math. Soc. **22** (1981), 25-43.

[9] A. K. Katsaras, The strict topology in non-Archimedean vector-valued function spaces, Proc. Kon. Ned. Akad. Wet. A **87 (2)** (1984), 189-201.

[10] A. K. Katsaras, Strict topologies in non-Archimedean functioin spaces, Intern. J. Math. and Math. Sci. **7 (1)**, (1984), 23-33.

[11] A. K. Katsaras, On the strict topology in non-Archimedean spaces of continuous functions, Glasnik Mat. vol **35 (55)** (2000), 283-305.

[12] A. K. Katsaras, Separable measures and strict topologies on spaces of non-Archimedean valued functions, in : P-adic Numbers in Number Theory, Analytic Geometry and Functional Analysis, edided by S. Caenepeel, Bull. Belgian Math., (2002), 117-139.

[13] A. K. Katsaras, Strict topologies and vector measures on non-Archimedean spaces, Cont. Math. vol. **319** (20053), 109-129.

[14] A. K. Katsaras, Non-Archimedean integration and strict rtopologies, Cont. Math. vol. **384** (2005), 111-144.

[15] A. K. Katsaras, Bornological spaces of non-Archimedean valued functions, Proc. Kon. Ned. Akad. Wet. **A 90** (1987), 41-50.

[16] A. K. Katsaras The non-Archimedean Grothendieck's completeness theorem, Bull. Inst. Math. Acad. Sinica **19** (1991), 351-354.

[17] A. K. Katsaras and A. Beloyiannis, Tensor products in non-Archimedean weighted spaces of continuous functions, Georgian J. Math. Vol. **6**, no 1 (1994), 33-44.

[18] A. K. Katsaras and C. G. Petalas, P-adic spaces with strict topologies as topological algebras, Cont. Math. vol. **319** (2003), 131-138.

[19] C. Perez-Garcia, P-adic Ascoli theorems and compactoid polynomials, Indag. Math. N. S. **3 (2)** (1993), 203-210.

[20] J. B. Prolla, Approximartion of vector-valued functions, North Holland Publ. Co., Amsterdam, New York, Oxforfd, 1977.

[21] W. H. Schikhof, Locally convex spaces over non-spherically complete fields I, II, Bull. Soc. Math. Belg., Ser. B, **38** (1986), 187-224.

[22] A. C. M. van Rooij, Non-Archimedean Functional Analysis, New York and Bassel, Marcel Dekker, 1978.

Vlachos extended their result ([13]):

Theorem 2 (Vlachos) *Let M be an odd n-dimensional compact oriented submanifold of the unit sphere S^{n+l} with mean curvature H. Assume that the Ricci curvature satisfies*

$$Ric > \frac{n(n-3)}{n-1} + \frac{n^2(n-3)}{(n-1)^2}H^2 + \frac{n(n-3)}{(n-1)^2}H\sqrt{n^2H^2+n^2-1}$$

If $n > 3$, then M is homeomorphic to a sphere; If $n = 3$, then M is diffeomorphic to a space form of positive sectional curvature.

In this note we extend the theorem above for even dimensional submanifolds of spheres.

Theorem *Let M be a simply connected $2m$-dimensional compact oriented submanifold of the unit sphere S^{2m+l} with nonnegative curvature operator. Assume that the Ricci curvature satisfies*

$$Ric > \frac{2m(2m-3)}{2m-1} + \frac{4m^2(2m-3)}{(2m-1)^2}H^2 + \frac{2m(2m-3)}{(2m-1)^2}H\sqrt{4m^2H^2+2m^2-1}$$

If $m > 2$, then M is homeomorphic to S^{2m} or $S^m \times S^m$. If $m = 2$, then M is homeomorphic to $\mathbb{C}P^2$, S^4, or $S^2 \times S^2$. Here H denotes the mean curvature.

We thank T. Hasanis for pointing out the proper expression of our theorem and T. Vlachos for bringing to our attention a relative result of B. Chow and D. Yang ([3]) and Z. Hu and S. Zhai ([8]).

2 Proof of our theorem

Let M denote an n-dimensional compact oriented Riemannian manifold equipped with a metric $<-,->$. Let $\mathcal{VF}(M)$ and $\mathcal{D}(M)$ denote the differentiable vector fields and differentiable functions on M respectively. Then $\mathcal{VF}(M)$ is a $\mathcal{D}(M)$-module and $\mathcal{VF}(M)^*$ denotes the $\mathcal{D}(M)$ dual of $\mathcal{VF}(M)$. Any element χ of $\mathcal{VF}(M)$ can be identified with the tensor $\chi : \mathcal{VF}(M) \to \mathcal{D}(M)$ given by $\chi(\gamma) =< \chi, \gamma >$. Thus $\chi \in \mathcal{VF}(M)^*$. Let ∇ denote the associated Levi-Civita connection $\nabla : \mathcal{VF}(M) \times \mathcal{VF}(M) \to \mathcal{VF}(M)$, then $\nabla_\chi(\gamma) \in \mathcal{VF}(M)^*$. For any pair $(\chi, \gamma) \in \mathcal{VF}(M) \times \mathcal{VF}(M)$ the Riemannian curvature $R(\chi, \gamma) : \mathcal{VF}(M) \to \mathcal{VF}(M)$ given by $R(\chi, \gamma)(\zeta)$ is an element of $\mathcal{E}nd_{\mathcal{D}(M)}(\mathcal{VF}(M))$ or $R(\chi, \gamma) : \mathcal{VF}(M) \times \mathcal{VF}(M) \to \mathcal{D}(M)$, i.e. $R(\chi, \gamma) \in (\mathcal{VF}(M) \times \mathcal{VF}(M))^*$. Thus at each point $p \in M$, $R : \mathcal{VF}(M) \times \mathcal{VF}(M) \to (\mathcal{VF}(M) \times \mathcal{VF}(M))^*$ defines an endomorphism of the space of bilinear antisymmetric forms on T_pM. This operator is denoted by $\rho_p : \Lambda^2(T_pM) \to \Lambda^2(T_pM)$ and is called the curvature operator at p.

Let $T_pM =< e_i | i = 1, ..., n >$ be an orthonormal basis and $T_pM^* =< e_i^* | i = 1, ..., n >$ its dual space. The diagonal elements of the matrix associated with ρ_p with respect to the basis $\{e_i^* \wedge e_j^* | i, j = 1, ..., n\}$ describe the sectional